Active Calculus & Mathematical Modeling
Activities and Voting Questions
Carroll College
MA 121

Carroll College Mathematics Department

Last Update: July 14, 2017
To The Student

This packet is NOT your textbook. What you will find here are

• **Preview Activities**
  These questions are designed to be done before class. They are meant to give you an accessible preview of the material covered in a section of the text.

• **Activities**
  These questions are meant to teach and re-enforce material presented in a section of the book. Many of these questions will be discussed during class time.

• **Voting Questions**
  These are multiple choice and true/false questions that will be used with the classroom clicker technology. Think of these as concept questions that address new definitions, new techniques, common misconceptions, and good review.

You should bring this booklet to class every day, but again remember that

**THIS BOOKLET IS NOT THE TEXTBOOK FOR THE CLASS.**

Instead, this booklet contains supplemental material.

To access the full textbook go to the Moodle page for your class or [www.carroll.edu/mathtexts/](http://www.carroll.edu/mathtexts/).

I’ll wait while you go to that website and have a look

... OK. Now that you’ve looked at the text you’ll know where to go to read more about the material being presented in class. If you want to print out the text then the costs are on you.
Contents

0 Preliminaries 7
  0.1 Functions, Slope, and Lines ........................................... 7
  0.2 Exponential Functions ................................................. 21
  0.3 Transformations of Functions ......................................... 26
  0.4 Logarithmic Functions ................................................. 40
  0.5 Trigonometric Functions ................................................ 53
  0.6 Powers, Polynomials, and Rational Functions ...................... 64

1 Understanding the Derivative 77
  1.1 How do we measure velocity? .......................................... 77
  1.2 The notion of limit ...................................................... 85
  1.3 The derivative of a function at a point ............................... 93
  1.4 The derivative function ................................................. 99
  1.5 Interpreting the derivative and its units ............................. 107
  1.6 The second derivative ................................................ 113
  1.7 Limits, Continuity, and Differentiability .............................. 123
  1.8 The Tangent Line Approximation ...................................... 131

2 Computing Derivatives 139
  2.1 Elementary derivative rules ............................................ 139
  2.2 The sine and cosine functions ........................................ 152
  2.3 The product and quotient rules ....................................... 158
  2.4 Derivatives of other trigonometric functions ....................... 167
  2.5 The chain rule .......................................................... 172
2.6 Derivatives of Inverse Functions ................................................. 180
2.7 Derivatives of Functions Given Implicitly ................................. 186
2.8 Using Derivatives to Evaluate Limits ....................................... 191
2.9 More on Limits ..................................................................... 199

3 Using Derivatives .................................................................. 205
3.1 Using derivatives to identify extreme values of a function ........ 205
3.2 Using derivatives to describe families of functions .................. 212
3.3 Global Optimization .............................................................. 218
3.4 Introduction to Sensitivity Analysis ......................................... 224
3.5 Applied Optimization ............................................................. 228
3.6 Related Rates ...................................................................... 234
3.7 Global Optimization ............................................................. 241
Chapter 0

Preliminaries

0.1 Functions, Slope, and Lines

**Preview Activity 0.1.** This is the first Preview Activity in this text. Your job for this activity is to get to know the textbook.

(a) Where can you find the full textbook?

(b) What chapters of this text are you going to cover this semester. Have a look at your syllabus!

(c) What are the differences between Preview Activities, Activities, Examples, Exercises, Voting Questions, and WeBWork? Which ones should you do before class, which ones will you likely do during class, and which ones should you be doing after class?

(d) What materials in this text would you use to prepare for an exam and where do you find them?

(e) What should you bring to class every day?
Activity 0.1.

The graph of a function $f(x)$ is shown in the plot below.

(a) What is the domain of $f(x)$?
(b) Approximate the range of $f(x)$.
(c) What are $f(0)$, $f(1)$, $f(3)$, $f(4)$, and $f(5)$?
Activity 0.2.

Find the equation of the line with the given information.

(a) The line goes through the points $(-2, 5)$ and $(10, -1)$.

(b) The slope of the line is $\frac{3}{5}$ and it goes through the point $(2, 3)$.

(c) The $y$-intercept of the line is $(0, -1)$ and the slope is $-\frac{2}{3}$. 
Activity 0.3.

An apartment manager keeps careful record of the rent that he charges as well as the number of occupied apartments in his complex. The data that he has is shown in the table below.

<table>
<thead>
<tr>
<th>Monthly Rent</th>
<th>$650</th>
<th>$700</th>
<th>$750</th>
<th>$800</th>
<th>$850</th>
<th>$900</th>
</tr>
</thead>
<tbody>
<tr>
<td>Occupied Apartments</td>
<td>203</td>
<td>196</td>
<td>189</td>
<td>182</td>
<td>175</td>
<td>168</td>
</tr>
</tbody>
</table>

(a) Just by doing simple arithmetic justify that the function relating the number of occupied apartments and the rent is linear.

(b) Find the linear function relating the number of occupied apartments to the rent.

(c) If the rent were to be increased to $1000, how many occupied apartments would the apartment manager expect to have?

(d) At a $1000 monthly rent what net revenue should the apartment manager expect?
Activity 0.4.

Write the equation of the line with the given information.

(a) Write the equation of a line parallel to the line $y = \frac{1}{2}x + 3$ passing through the point $(3, 4)$.

(b) Write the equation of a line perpendicular to the line $y = \frac{1}{2}x + 3$ passing through the point $(3, 4)$.

(c) Write the equation of a line with $y$-intercept $(0, -3)$ that is perpendicular to the line $y = -3x - 1$. 

\[<\]
Voting Questions

0.1.1 In the given equation, is $y$ a function of $x$?

\[ y = x + 2 \]

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.2 In the given equation, is $y$ a function of $x$?

\[ x + y = 5 \]

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.3 In the given equation, is $y$ a function of $x$?

\[ x^3 + y = 5 \]

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.4 In the given equation, is $y$ a function of $x$?

\[ x^2 + y^2 = 5 \]

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.5 The set of points \((x, y)\) which satisfy the equation \((x - 1)^2 + (y + 3)^2 = 5^2\) can be represented via a mathematical function relating the \(x\) and \(y\) variables.

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.1.6 Does the table represent a function, \(y = f(x)\)?

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.7 Does the table represent a function, \(y = f(x)\)?

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.8 Does this sentence describe a function? Wanda is two years older than I am.

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident
0.1.9 The rule which assigns to each college student (at this exact point in time) a number equal
to the number of college credits completed by that student is a function.

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.1.10 The rule which assigns to each car (at this exact point in time) the names of every person
that has driven that car is a function.

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.1.11 Could this table represent a linear function?

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.12 Could this table represent a linear function?

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-12</td>
<td>-9</td>
<td>-6</td>
<td>-3</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident
0.1.13 Could this table represent a linear function?

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.14 Could this table represent a linear function?

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>10</td>
<td>9</td>
<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.15 True or False? All linear functions are examples of direct proportionality.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

0.1.16 Find the domain of the function $f(x) = \frac{1}{x-2}$.

(a) $x = 2$
(b) $x \neq 2$
(c) $x < 2$
(d) all real numbers

0.1.17 Find the domain of the function $g(t) = \frac{2+t}{\sqrt{4-t}}$.

(a) $t > 7$
(b) $t \geq 7$
0.1. FUNCTIONS, SLOPE, AND LINES

(c) \( t = 7 \)
(d) all real numbers

0.1.18 Which of the following functions has its domain identical with its range?

(a) \( f(x) = x^2 \)
(b) \( g(x) = \sqrt{x} \)
(c) \( h(x) = x^4 \)
(d) \( i(x) = |x| \)

0.1.19 The slope of the line connecting the points (1,4) and (3,8) is

(a) \(-\frac{1}{2}\)
(b) \(-2\)
(c) \(\frac{1}{2}\)
(d) 2

0.1.20 Which one of these lines has a different slope than the others?

(a) \( y = 3x + 2 \)
(b) \(3y = 9x + 4\)
(c) \(3y = 3x + 6\)
(d) \(2y = 6x + 4\)

0.1.21 The graph below represents which function?

(a) \( y = 6x + 6 \)
(b) \( y = -3x + 6 \)
(c) \( y = -3x + 2 \)
(d) \( y = -x + 6 \)
(e) \( y = 6x - 2 \)
(f) \( y = x - 2 \)
0.1.22 Which of the following functions is not increasing?

(a) The elevation of a river as a function of distance from its mouth
(b) The length of a single strand of hair as a function of time
(c) The height of a person from age 0 to age 80
(d) The height of a redwood tree as a function of time

0.1.23 Which of these graphs does not represent $y$ as a function of $x$?

![Plot (a)](image-a)

![Plot (b)](image-b)

![Plot (c)](image-c)

![Plot (d)](image-d)

0.1.24 Calculate the average rate of change of the function $f(x) = x^2$ between $x = 1$ and $x = 3$.

(a) 8
(b) 4
0.1.25 The EPA reports the total amount of Municipal Solid Waste (MSW), otherwise known as garbage, produced in the U.S. for the years 2005 through 2009:

<table>
<thead>
<tr>
<th>Year</th>
<th>Millions of tons</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005</td>
<td>252.4</td>
</tr>
<tr>
<td>2006</td>
<td>251.3</td>
</tr>
<tr>
<td>2007</td>
<td>255</td>
</tr>
<tr>
<td>2008</td>
<td>249.6</td>
</tr>
<tr>
<td>2009</td>
<td>243</td>
</tr>
</tbody>
</table>

(source: http://www.epa.gov/osw/nonhaz/municipal/)

What are the appropriate units for the average rate of change in the amount of garbage produced between any two given years?

(a) millions of tons
(b) tons
(c) millions of tons per year
(d) tons per year

0.1.26 The EPA reports the total amount of Municipal Solid Waste (MSW), otherwise known as garbage, produced in the U.S. for the years 2005 through 2009:

<table>
<thead>
<tr>
<th>Year</th>
<th>Millions of tons</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005</td>
<td>252.4</td>
</tr>
<tr>
<td>2006</td>
<td>251.3</td>
</tr>
<tr>
<td>2007</td>
<td>255</td>
</tr>
<tr>
<td>2008</td>
<td>249.6</td>
</tr>
<tr>
<td>2009</td>
<td>243</td>
</tr>
</tbody>
</table>

(source: http://www.epa.gov/osw/nonhaz/municipal/)

What is the average rate of change in the amount of MSW produced from 2005 to 2007?

(a) 2.6 million tons per year
(b) 2.6 million tons
(c) 1.3 million tons
(d) 1.3 million tons per year

0.1.27 The EPA reports the total amount of Municipal Solid Waste (MSW), otherwise known as garbage, produced in the U.S. for the years 2005 through 2009:
What is the average rate of change in the amount of MSW produced from 2007 to 2009?

(a) $-6$ million tons per year
(b) $6$ million tons per year
(c) $-12$ million tons per year
(d) $12$ million tons per year

0.1.28 Find the difference quotient \( \frac{f(x+h)-f(x)}{h} \) for the function \( f(x) = 2x^2 - x + 3 \). Simplify your answer.

(a) \( \frac{2h^2-h+3}{h} \)
(b) \( 2h - 1 \)
(c) \( \frac{4xh+2h^2-2x+h+6}{h} \)
(d) \( 4x + 2h - 1 \)

0.1.29 When the temperature is $0^\circ C$ it is $32^\circ F$ and when it is $100^\circ C$ it is $212^\circ F$. Use these facts to write a linear function to convert any temperature from Celsius to Fahrenheit.

(a) \( C(F) = \frac{5}{9}F - \frac{160}{9} \)
(b) \( F(C) = C + 32 \)
(c) \( F(C) = \frac{9}{5}C - \frac{160}{9} \)
(d) \( F(C) = \frac{9}{5}C + 32 \)

0.1.30 Let \( f(x) = 1 + 4x^2 \). True or False: \( f(\frac{1}{2}) = \frac{f(1)}{f(2)} \).

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.1.31 Let \( f(x) = 1 + 4x^2 \). True or False: \( f(a + b) = f(a) + f(b) \).
(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.1.32 Let \( f(x) = \frac{1}{x+2} \). Find a value of \( x \) so that \( f(x) = 6 \)

(a) \(-\frac{11}{6}\)
(b) \(\frac{13}{6}\)
(c) \(\frac{1}{8}\)
(d) none of the above

0.1.33 True or False: \( \sqrt{x^2} = x \).

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.
0.2 Exponential Functions

Preview Activity 0.2. Suppose that the populations of two towns are both growing over time. The town of Exponentia is growing at a rate of 2% per year, and the town of Lineola is growing at a rate of 100 people per year. In 2014, both of the towns have 2,000 people.

(a) Complete the table for the population of each of these towns over the next several years.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponentia</td>
<td>2000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lineola</td>
<td>2000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Write a linear function for the population of Lineola. Interpret the slope in the context of this problem.

(c) The ratio of successive populations for Exponentia should be equal. For example, dividing the population in 2015 by that of 2014 should give the same ratio as when the population from 2016 is divided by the population of 2015. Find this ratio. How is this ratio related to the 2% growth rate?

(d) Based on your data from part (a) and your ratio in part (c), write a function for the population of Exponentia.

(e) When will the population of Exponentia exceed that of Lineola?
Activity 0.5.

Consider the exponential functions plotted in Figure 1

(a) Which of the functions have common ratio $r > 1$?

(b) Which of the functions have common ratio $0 < r < 1$?

(c) Rank each of the functions in order from largest to smallest $r$ value.

Figure 1: Exponential growth and decay functions
Activity 0.6.

A sample of $Ni^{56}$ has a half-life of 6.4 days. Assume that there are 30 grams present initially.

(a) Write a function describing the number of grams of $Ni^{56}$ present as a function of time. Check your function based on the fact that in 6.4 days there should be 50% remaining.

(b) What percent of the substance is present after 1 day?

(c) What percent of the substance is present after 10 days?
Activity 0.7.

Uncontrolled geometric growth of the bacterium *Escherichia coli* (*E. Coli*) is the theme of the following quote taken from the best-selling author Michael Crichton’s science fiction thriller, *The Andromeda Strain*:

“The mathematics of uncontrolled growth are frightening. A single cell of the bacterium *E. coli* would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of *E. coli* could produce a super-colony equal in size and weight to the entire planet Earth.”

(a) Write an equation for the number of *E. coli* cells present if a single cell of *E. coli* divides every 20 minutes.

(b) How many *E. coli* would there be at the end of 24 hours?

(c) The mass of an *E. coli* bacterium is $1.7 \times 10^{-12}$ grams, while the mass of the Earth is $6.0 \times 10^{27}$ grams. Is Michael Crichton’s claim accurate? Approximate the number of hours we should have allowed for this statement to be correct?
Voting Questions

0.2.1 The following table shows the net sales at Amazon.com from 2003 to 2010 (source: Amazon.com quarterly reports):

<table>
<thead>
<tr>
<th>Year</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Billions of dollars</td>
<td>$5.26</td>
<td>$6.92</td>
<td>$8.49</td>
<td>$10.72</td>
<td>$14.84</td>
<td>$19.15</td>
<td>$24.51</td>
<td>$34.21</td>
</tr>
</tbody>
</table>

If the net sales are modeled using an exponential function $S(t) = a \cdot b^t$, where $S$ is the net sales in billions of dollars, and $t$ is the number of years after 2003, which of the following is an appropriate value for $a$?

(a) 34.21
(b) 1
(c) 1.31
(d) 5.26

0.2.2 Which is better at the end of one year: An account that pays 8% annual interest compounded quarterly or an account that pays 7.95% interest compounded continuously?

(a) 8% quarterly
(b) 7.95% continuously
(c) They are the same.
(d) There is no way to tell.

0.2.3 Caffeine leaves the body at a continuous rate of 17% per hour. How much caffeine is left in the body 8 hours after drinking a cup of coffee containing 100 mg of caffeine?

(a) 389.62 mg
(b) 22.52 mg
(c) 25.67 mg
(d) There is no way to tell.

0.2.4 Caffeine leaves the body at a continuous rate of 17% per hour. What is the hourly growth factor?

(a) .156
(b) .17
(c) .844
(d) There is no way to tell.
0.3 Transformations of Functions

Preview Activity 0.3. The goal of this activity is to explore and experiment with the function

\[ F(x) = Af(B(x - C)) + D. \]

The values of \( A, B, C, \) and \( D \) are constants and the function \( f(x) \) will be henceforth called the parent function. To facilitate this exploration, use the applet located at http://www.geogebratube.org/student/m93018.

(a) Let’s start with a simple parent function: \( f(x) = x^2 \).

(1) Fix \( B = 1, C = 0, \) and \( D = 0 \). Write a sentence or two describing the action of \( A \) on the function \( F(x) \).

(2) Fix \( A = 1, B = 1, \) and \( D = 0 \). Write a sentence of two describing the action of \( C \) on the function \( F(x) \).

(3) Fix \( A = 1, B = 1, \) and \( C = 0 \). Write a sentence of two describing the action of \( D \) on the function \( F(x) \).

(4) Fix \( A = 1, C = 0, \) and \( D = 0 \). Write a sentence of two describing the action of \( B \) on the function \( F(x) \).

(b) In part (a) you have made conjectures about what \( A, B, C, \) and \( D \) do to a parent function graphically. Test your conjectures with the functions \( f(x) = |x| \) (typed abs (x)), \( f(x) = x^3 \), \( f(x) = \sin(x) \), \( f(x) = e^x \) (typed exp (x)), and any other function you find interesting.

\[ \blacklozenge \]
Activity 0.8.

Consider the function $f(x)$ displayed in Figure 2.

(a) Plot $g(x) = -f(x)$ and $h(x) = f(x) - 1$.

(b) Define the function $k(x) = -f(x) - 1$. Does it matter which order you complete the transformations from part (a) to result in $k(x)$? Plot the functions resulting from doing the two transformation in part (a) in opposite orders. Which of these functions is $k(x)$?

Figure 2: Function transformation for Activity 0.8
Activity 0.9.

(a) Let \( f(x) = x^2 \) and \( g(x) = x + 8 \). Find the following:

\[
\begin{align*}
  f(g(3)) &= \\
  g(3) &= \\
  f(3) &= \\
  g(f(x)) &= \\
  f(x)g(x) &= \\
\end{align*}
\]

(b) Now let \( f(x) \) and \( g(x) \) be defined as in the table below. Use the data in the table to find the following compositions.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
  f(-3) &= \\
  g(3) &= \\
  f(g(-3)) &= \\
  f(g(f(-3))) &= \\
\end{align*}
\]

(c) Now let \( f(x) \) and \( g(x) \) be defined as in the plots below. Use the plots to find the following compositions.

\[
\begin{align*}
  f(1) &= \\
  g(2) &= \\
  g(f(1)) &= \\
  f(g(1)) &= \\
  g(f(f(0))) &= \\
\end{align*}
\]
Activity 0.10.
(a) Based on symmetry alone, is \( f(x) = x^2 \) an even or an odd function?
(b) Based on symmetry alone, is \( g(x) = x^3 \) an even or an odd function?
(c) Find \( f(-x) \) and \( g(-x) \) and make conjectures to complete these sentences:
   - If a function \( f(x) \) is \textit{even} then \( f(-x) = \underline{\phantom{0}} \).
   - If a function \( f(x) \) is \textit{odd} then \( f(-x) = \underline{\phantom{0}} \).

Explain why the composition \( f(-x) \) is a good test for symmetry of a function.
(d) Classify each of the following functions as even, odd, or neither.
\[
h(x) = \frac{1}{x}, \quad j(x) = e^x, \quad k(x) = x^2 - x^4, \quad n(x) = x^3 + x^2.
\]
(e) Each figure below shows only half of the function. Draw the left half so \( f(x) \) is even. Draw the left half so \( g(x) \) is odd. Draw the left half so \( h(x) \) is neither even nor odd.
Activity 0.11.

(a) Find the inverse of each of the following functions by interchanging the $x$ and $y$ and solving for $y$. Be sure to state the domain for each of your answers.

\[
y = \sqrt{x - 1}, \quad y = -\frac{1}{3}x + 1, \quad y = \frac{x + 4}{2x - 5}
\]

(b) Verify that the functions $f(x) = 3x - 2$ and $g(x) = \frac{2}{3} + \frac{2}{3}$ are inverses of each other by computing $f(g(x))$ and $g(f(x))$. 

\[\triangleright\]
Voting Questions

0.3.1 The functions \( f \) and \( g \) have values given in the table below. What is the value of \( f(g(0)) \)?

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

(a) -2  
(b) -1  
(c) 0  
(d) 1  
(e) 2

0.3.2 The functions \( f \) and \( g \) have values given in the table below. If \( f(g(x)) = 1 \), then what is \( x \)?

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

(a) -2  
(b) -1  
(c) 0  
(d) 1  
(e) 2

0.3.3 The graphs of \( f \) and \( g \) are shown in the figure below. Estimate the value of \( g(f(3)) \).

(a) -1  
(b) 0  
(c) 1  
(d) 2  
(e) 3  
(f) 5
0.3.4 The graphs of \( f \) and \( g \) are shown in the figure below. Estimate the value of \( f(g(2)) \).

(a) -1  
(b) 0  
(c) 1  
(d) 2  
(e) 3  
(f) 5

0.3.5 If \( P = f(t) = 3 + 4t \), find \( f^{-1}(P) \).

(a) \( f^{-1}(P) = 3 + 4P \)  
(b) \( f^{-1}(P) = \frac{P-3}{4} \)  
(c) \( f^{-1}(P) = \frac{P-4}{3} \)  
(d) \( f^{-1}(P) = 4(P + 3) \)  
(e) \( f^{-1}(P) = \frac{P+3}{4} \)

0.3.6 Which of these functions has an inverse?

Plot (a)  
Plot (b)
0.3.7 The following is a graph of $f(x)$. Which graph below is the inverse?

(a) (a) only
(b) (b) only
(c) (c) only
(d) (d) only
(e) (a) and (b)
(f) (b) and (c)
0.3.8 Given that \( f(x) = \sqrt{\frac{x^3 - 72}{800}} \), find \( f \circ f^{-1}(437) \).

(a) 104, 316.73  
(b) 1671.2  
(c) 437  
(d) 10.08

0.3.9 If \( f(x) = \frac{x}{x^2 + 1} \), what is \( f^{-1} \circ f(-2) \)?

(a) \( \frac{2}{5} \)  
(b) \( \frac{2}{3} \)
0.3. TRANSFORMATIONS OF FUNCTIONS

0.3.10 If \((4, -2)\) is a point on the graph of \(y = f(x)\), which of the following points is on the graph of \(y = f^{-1}(x)\)?

(a) \((-2, 4)\)
(b) \((-4, 2)\)
(c) \((\frac{1}{4}, -\frac{1}{2})\)
(d) \((-\frac{1}{4}, \frac{1}{2})\)

0.3.11 Find the inverse of \(f(x) = \frac{1}{x}\).

(a) \(f^{-1}(x) = \frac{x}{1}\)
(b) \(f^{-1}(x) = x\)
(c) \(f^{-1}(x) = \frac{1}{x}\)
(d) \(f^{-1}(x) = xy\)

0.3.12 A function is given in Figure 1.10 below. Which one of the other graphs could be a graph of \(f(x + h)\)?
0.3.13 How is the graph of $y = 2^{x-1} + 3$ obtained from the graph of $y = 2^x$?

(a) Move 1 down and 3 right
(b) Move 1 left and 3 up
(c) Move 1 up and 3 right
(d) Move 1 right and 3 up

0.3.14 The function $f(x)$ goes through the point $A$ with coordinates (2,3). $g(x) = 2f\left(\frac{1}{3}x - 2\right) + 4$. What are the coordinates of point $A$ in the function $g(x)$?

(a) (4, 10)
(b) (4, $-\frac{5}{2}$)
0.3. TRANSFORMATIONS OF FUNCTIONS

(c) (12, 10)
(d) \((-\frac{4}{3}, 10)\)
(e) \((-\frac{4}{3}, -\frac{5}{2})\)

0.3.15 The point (4, 1) is on the graph of a function \(f\). Find the corresponding point on the graph of \(y = f(x - 2)\).

(a) (6, 1)
(b) (2, 1)
(c) (4, 3)
(d) (4, -1)

0.3.16 The point (6, 1) is on the graph of a function \(f\). Find the corresponding point on the graph of \(y = f(2x)\).

(a) (12, 1)
(b) (3, 1)
(c) (6, 2)
(d) (6, \frac{1}{2})

0.3.17 Given the graph of a function \(f(x)\), what sequence of activities best describes the process you might go through to graph \(g(x) = 5f(-x)\)?

(a) Expand the graph by a factor of 5, then reflect it across the \(y\)-axis.
(b) Expand the graph by a factor of 5, then reflect it across the \(x\)-axis.
(c) Reflect the graph across the \(y\)-axis, then expand it by a factor of 5.
(d) Reflect the graph across the \(x\)-axis, then expand it by a factor of 5.
(e) More than 1 of the above.
(f) None of the above.

0.3.18 Given the graph of a function \(f(x)\), what sequence of activities best describes the process you might go through to graph \(g(x) = -f(x) + 2\)?

(a) Move the graph up 2 units, then reflect it across the \(x\)-axis.
(b) Move the graph up 2 units, then reflect it across the \(y\)-axis.
(c) Reflect the graph across the $y$-axis, then move it up by 2 units.
(d) Reflect the graph across the $x$-axis, then move it up 2 units.
(e) More than 1 of the above.
(f) None of the above.

0.3.19 Take the function $f(x)$ and “Shift the function right $h$ units. Reflect the result across the $y$-axis, then reflect the result across the $x$-axis. Finally shift the result up $k$ units.” The end result is:

(a) $f(x + h) + k$
(b) $f(x - h) + k$
(c) $-f(-x - h) + k$
(d) $-f(-x + h) + k$

0.3.20 Given $f(x) = x + 1$ and $g(x) = 3x^2 - 2x$, what is the composition $g(f(x))$.

(a) $3x^2 - 2x + 1$
(b) $(3x^2 - 2x)(x + 1)$
(c) $3x^2 + 4x + 1$
(d) $3(x + 1)^2 - 2x$

0.3.21 Write $h(x) = e^{3x/2}$ as a composition of functions: $f(g(x))$. $f(x) =$ ____________, $g(x) =$ ____________.

(a) $e^x, 3x/2$
(b) $3x/2, e^x$
(c) $x, e^{3x/2}$
(d) $x/2, 3e^x$

0.3.22 If $f(x) = x^2 + 6$ and $g(x) = x - 3$, what is $f \circ g(x)$?

(a) $x^2 + 3$
(b) $x^2 - 6x + 15$
(c) $x^2 - 3$
(d) $x^3 - 3x^2 + 6x - 18$
0.3.23 Which of the following functions IS invertible?

(a) \( f(x) = -x^4 + 7 \)
(b) \( g(x) = e^{3x/2} \)
(c) \( h(x) = \cos(x) \)
(d) \( k(x) = |x| \)

0.3.24 Let \( f(x) = x - 2 \) and \( g(x) = 3 - x^2 \). Find \( g(f(2)) \).

(a) -3
(b) 0
(c) 3
(d) 2

0.3.25 If \( P = f(t) = 3 + 4t \), find \( f^{-1}(7) \).

(a) 31
(b) \( \frac{1}{7} \)
(c) 0
(d) 1

0.3.26 Let \( f(x) = x^2 \) and \( g(x) = x + 2 \). True or false? The domain of the function \( \frac{f}{g} \) is \( \mathbb{R} \), all real numbers.

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.3.27 Let \( f(x) = x^2 - 4 \) and \( g(x) = \sqrt{x} \). Find \( (g \circ f)(x) \) and the domain of \( g \circ f \).

(a) \( \sqrt{x^2 - 4} \); Domain: \((-\infty, -2] \cup [2, \infty)\)
(b) \( x - 4 \); Domain: \( \mathbb{R} \)
(c) \( x - 4 \); Domain: \([0, \infty)\)
(d) \( \sqrt{x^2 - 4} \); Domain: \([0, \infty)\)
(e) \( \sqrt[3]{(x^2 - 4)} \); Domain: \([0, \infty)\)
0.4 Logarithmic Functions

Preview Activity 0.4. Carbon-14 ($^{14}\text{C}$) is a radioactive isotope of carbon that occurs naturally in the Earth’s atmosphere. During photosynthesis, plants take in $^{14}\text{C}$ along with other carbon isotopes, and the levels of $^{14}\text{C}$ in living plants are roughly the same as atmospheric levels. Once a plant dies, it no longer takes in any additional $^{14}\text{C}$. Since $^{14}\text{C}$ in the dead plant decays at a predictable rate (the half-life of $^{14}\text{C}$ is approximately 5,730 years), we can measure $^{14}\text{C}$ levels in dead plant matter to get an estimate on how long ago the plant died. Suppose that a plant has 0.02 milligrams of $^{14}\text{C}$ when it dies.

(a) Write a function that represents the amount of $^{14}\text{C}$ remaining in the plant after $t$ years.

(b) Complete the table for the amount of $^{14}\text{C}$ remaining $t$ years after the death of the plant.

(c) Suppose our plant died sometime in the past. If we find that there are 0.014 milligrams of $^{14}\text{C}$ present in the plant now, estimate the age of the plant to within 50 years.
Activity 0.12.

Use the definition of a logarithm along with the properties of logarithms to answer the following.

(a) Write the exponential expression $8^{1/3} = 2$ as a logarithmic expression.
(b) Write the logarithmic expression $\log_2 \frac{1}{32} = -5$ as an exponential expression.
(c) What value of $x$ solves the equation $\log_2 x = 3$?
(d) What value of $x$ solves the equation $\log_2 4 = x$?
(e) Use the laws of logarithms to rewrite the expression $\log (x^3y^5)$ in a form with no logarithms of products, quotients, or powers.
(f) Use the laws of logarithms to rewrite the expression $\log \left( \frac{x^{15}y^{20}}{z^4} \right)$ in a form with no logarithms of products, quotients, or powers.
(g) Rewrite the expression $\ln(8) + 5 \ln(x) + 15 \ln(x^2 + 8)$ as a single logarithm.
Activity 0.13.
Solve each of the following equations for \( t \), and verify your answers using a calculator.

(a) \( \ln t = 4 \)
(b) \( \ln(t + 3) = 4 \)
(c) \( \ln(t + 3) = \ln 4 \)
(d) \( \ln(t + 3) + \ln(t) = \ln 4 \)
(e) \( e^t = 4 \)
(f) \( e^{t+3} = 4 \)
(g) \( 2e^{t+3} = 4 \)
(h) \( 2e^{3t+2} = 3e^{t-1} \)
Activity 0.14.
Consider the following equation:

\[ 7^x = 24 \]

(a) How many solutions should we expect to find for this equation?
(b) Solve the equation using the log base 7.
(c) Solve the equation using the log base 10.
(d) Solve the equation using the natural log.
(e) Most calculators have buttons for \( \log_{10} \) and \( \ln \), but none have a button for \( \log_7 \). Use your previous answers to write a formula for \( \log_7 x \) in terms of \( \log x \) or \( \ln x \).
Activity 0.15.

(a) In the presence of sufficient resources the population of a colony of bacteria exhibits exponential growth, doubling once every three hours. What is the corresponding continuous (percentage) growth rate?

(b) A hot bowl of soup is served at a dinner party. It starts to cool according to Newton’s Law of Cooling so its temperature, \( T \) (measured in degrees Fahrenheit) after \( t \) minutes is given by

\[
T(t) = 65 + 186e^{-0.06t}.
\]

How long will it take from the time the food is served until the temperature is 120°F?

(c) The velocity (in ft/sec) of a sky diver \( t \) seconds after jumping is given by

\[
v(t) = 80 \left( 1 - e^{-0.2t} \right).
\]

After how many seconds is the velocity 75 ft/sec?
0.4. LOGARITHMIC FUNCTIONS

Voting Questions

0.4.1 A logarithmic function of the form \( f(x) = \log_a x \) will always pass through the point \((1, 0)\).

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.4.2 Which is a graph of \( y = \ln x \)?

![Graphs of logarithmic functions](image-url)
0.4.3 The graph below could be that of

(a) \( y = \ln x + \frac{1}{2} \)
(b) \( y = \ln x - \frac{1}{2} \)
(c) \( y = \ln (x + \frac{1}{2}) \)
(d) \( y = \ln (x - \frac{1}{2}) \)

0.4.4 Which equation matches this graph?

(a) \( y = b^x \) with \( b > 1 \)
(b) \( y = b^x \) with \( 0 < b < 1 \)
(c) \( y = \log_b x \) with \( b > 1 \)
(d) \( y = \log_b x \) with \( 0 < b < 1 \)

0.4.5 Which equation matches this graph?

(a) \( y = b^x \) with \( b > 1 \)
(b) \( y = b^x \) with \( 0 < b < 1 \)
(c) \( y = \log_b x \) with \( b > 1 \)
(d) \( y = \log_b x \) with \( 0 < b < 1 \)
0.4.6 Which of the following is a graph of $y = \log_2 x$?

0.4.7 Which of the following is a graph of $y = \log_{\frac{1}{2}} x$?
0.4.8 Which of the following functions have vertical asymptotes of $x = 3$?

(a) $y = \ln(x/3)$  
(b) $y = \ln(x - 3)$  
(c) $y = \ln(x + 3)$  
(d) $y = 3 \ln x$

0.4.9 $\log \left( \frac{M-N}{M+N} \right) =$

(a) $2 \log M$  
(b) $2 \log N$  
(c) $-2 \log N$
0.4. LOGARITHMIC FUNCTIONS

(d) \( \log(M - N) - \log(M + N) \)

0.4.10 If \( \log_{10}(x - a) = n \), then \( x = \)

(a) \( 10^{n+a} \)
(b) \( a + 10^n \)
(c) \( n + 10^a \)
(d) \( n + a^{10} \)

0.4.11 What is the exponential form of \( \log_r m = j \)?

(a) \( r^j = m \)
(b) \( j^r = m \)
(c) \( m^j = r \)
(d) \( r^m = j \)

0.4.12 What is the logarithmic form of \( k^p = d \)?

(a) \( \log_d k = p \)
(b) \( \log_k d = p \)
(c) \( \log_p d = p \)
(d) \( \log_k p = d \)

0.4.13 What is the value of \( \log_{11} 86 \)? (Calculators are allowed.)

(a) \( .4049 \)
(b) \( .5383 \)
(c) \( 1.8576 \)
(d) \( -2.0564 \)

0.4.14 What is \( 3 = \log_2 8 \) in exponential form?

(a) \( 2^8 = 3 \)
(b) \( 3^2 = 8 \)
(c) \( 8^3 = 2 \)
0.4. LOGARITHMIC FUNCTIONS

(d) \( 2^3 = 8 \)

0.4.15 What is \( k = \log_m q \) in exponential form?

(a) \( m^k = q \)
(b) \( k^q = m \)
(c) \( m^q = k \)
(d) \( q^m = k \)

0.4.16 What is \( 4^2 = 16 \) in logarithmic form?

(a) \( \log_2 4 = 16 \)
(b) \( \log_4 16 = 2 \)
(c) \( \log_4 2 = 16 \)
(d) \( \log_{16} 4 = 2 \)

0.4.17 What is \( 3^{-1} = \frac{1}{3} \) in logarithmic form?

(a) \( \log_3 (-1) = \frac{1}{3} \)
(b) \( \log_{-1} \frac{1}{3} = 3 \)
(c) \( \log_{\frac{1}{4}} 3 = -1 \)
(d) \( \log_3 \frac{1}{3} = -1 \)

0.4.18 What is the inverse of the following function:

\[ P = f(t) = 16 \ln(14t) \]

(a) \( f^{-1}(P) = \frac{1}{14} e^{\frac{16P}{16}} \)
(b) \( f^{-1}(P) = \frac{1}{14} e^{P/16} \)
(c) \( f^{-1}(P) = \frac{1}{14} \ln(P/16) \)
(d) \( f^{-1}(P) = \frac{\ln 16}{14} P \)
0.4.19 Solve for $x$ if $8y = 3e^x$.

(a) $x = \ln 8 + \ln 3 + \ln y$
(b) $x = \ln 3 - \ln 8 + \ln y$
(c) $x = \ln 8 + \ln y - \ln 3$
(d) $x = \ln 3 - \ln 8 - \ln y$

0.4.20 Solve for $x$ if $y = e + 2^x$

(a) $x = \frac{\ln y - 1}{\ln 2}$
(b) $x = \frac{\ln(y - 1)}{\ln 2}$
(c) $x = \frac{\ln y}{\ln 2} - 1$
(d) $x = \frac{\ln(y - e)}{\ln 2}$

0.4.21 Write the following expression using a single logarithmic function:

$$\ln(2x^3 + 1) + 5 \ln(3 - x) - \ln(6x^5 + 2x + 1).$$

(a) $\ln(-6x^5 + 2x^3 - 7x + 15)$
(b) $\ln[(2x^3 + 1)(15 - 5x)(-6x^5 - 2x - 1)]$
(c) $\ln\left(\frac{(2x^3 + 1)(3 - x)^5}{6x^5 + 2x + 1}\right)$
(d) $\ln\left(\frac{(2x^3 + 1)(15 - 5x)}{6x^5 + 2x + 1}\right)$

0.4.22 $\log\left(\frac{a^4 b^7}{c^5}\right) =$

(a) $\log(a^4) + \log(b^7) + \log(c^5)$
(b) $4 \log a + 7 \log b - 5 \log c$
(c) $28 \log ab - 5 \log c$
(d) $\frac{28}{5} (\log a + \log b - \log c)$
(e) None of the above

0.4.23 Simplify the following expression: $\ln\left(\frac{\sqrt{x^2 + 1}(x^3 - 4)}{(3x - 7)^2}\right)$. 


0.4. LOGARITHMIC FUNCTIONS

(a) $\frac{1}{2} \ln(x^2 + 1) + \ln(x^3 + 4) - 2 \ln(3x - 7)$

(b) $\ln\left(\frac{1}{2}(x^2 + 1)\right) + \ln(x^3 + 4) - 2 \ln(3x - 7)$

(c) $\ln(x^2 + 1) \ln(x^3 + 4) \ln(3x - 7)$

(d) $\ln[(x^2 + 1)(x^3 + 4)(3x - 7)]$

0.4.24 25 rabbits are introduced to an island, where they quickly reproduce and the rabbit population grows according to an exponential model $P(t) = P_0 e^{kt}$ so that the population doubles every four months. If $t$ is in months, what is the value of the continuous growth rate $k$?

(a) $k = \frac{1}{2} \ln 4$

(b) $k = \frac{1}{4} \ln 2$

(c) $k = \frac{1}{50} \ln \frac{4}{25}$

(d) $k = \frac{4}{25} \ln \frac{1}{50}$

(e) None of the above

0.4.25 Simplify $(\log_{16} 4) \left(\log_3 \frac{1}{9}\right)$.

(a) $\frac{16}{3}$

(b) $\frac{4}{9}$

(c) 1

(d) -1
0.5 Trigonometric Functions

Preview Activity 0.5. A tall water tower is swaying back and forth in the wind. Using an ultrasonic ranging device, we measure the distance from our device to the tower (in centimeters) every two seconds with these measurements recorded below.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (cm)</td>
<td>30.9</td>
<td>23.1</td>
<td>14.7</td>
<td>12.3</td>
<td>17.7</td>
<td>26.7</td>
<td>32.3</td>
<td>30.1</td>
<td>21.8</td>
<td>13.9</td>
<td>12.6</td>
</tr>
</tbody>
</table>

(a) Use the coordinate plane below to create a graph of these data points.

(b) What is the water tower’s maximum distance away from the device?

(c) What is the smallest distance measured from the tower to the device?

(d) If the water tower was sitting still and no wind was blowing, what would be the distance from the tower to the device? We call this the tower’s equilibrium position.

(e) What is the maximum distance that the tower moves away from its equilibrium position? We call this the amplitude of the oscillations.

(f) How much time does it take the water tower to sway back and forth in a complete cycle? We call this the period of oscillation.
Activity 0.16.

In this activity we will review the trigonometry of the special angles 0°, 30°, 45°, and their multiples.

(a) Use the fact that 180° is the same as \( \pi \) radians, convert each of the following angle measurements to radians.

<table>
<thead>
<tr>
<th>Degrees</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td>0</td>
<td>( \pi /6 )</td>
<td>( \pi /4 )</td>
<td>( \pi /3 )</td>
<td>( \pi /2 )</td>
<td>( 2\pi /3 )</td>
<td>( 3\pi /4 )</td>
<td>( 5\pi /6 )</td>
<td>( \pi )</td>
</tr>
</tbody>
</table>

(b) In part (a) of this problem there are several patterns that can help in remembering the radian conversions for certain angles. For example, you should have found that 30° converts to \( \pi /6 \) radians. Therefore, 60° should be twice \( \pi /6 \) which indeed it is: \( 60° = \pi /3 \) radians. What other similar patterns can you find? What is the minimum number of radian measures that you need to memorize?

(c) The sides of a 30° – 60° – 90° triangle follow well-known ratios. Consider the equilateral triangle on the left of the figure below. Fill in the rest of the sides and angles on the figure and use them to determine the trigonometric values of 30° and 60°.

<table>
<thead>
<tr>
<th>Angle (degrees)</th>
<th>Angle (radians)</th>
<th>Sine</th>
<th>Cosine</th>
<th>Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>60°</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) The sides of a 45° – 45° – 90° triangle also follow well-known ratios. Consider the isosceles triangle on the right of the figure below. Fill in the rest of the sides and angles on the figure and use them to determine the trigonometric values of 45°.

<table>
<thead>
<tr>
<th>Angle (degrees)</th>
<th>Angle (radians)</th>
<th>Sine</th>
<th>Cosine</th>
<th>Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(e) Finally, we can organize all of the information about the special right triangles on a well-known organizational tool: the unit circle. The \( x \) coordinate of each point is the cosine of the angle and the \( y \) coordinate of each point is the sine of the angle.
Activity 0.17.

Figure 3 gives us the voltage produced by an electrical circuit as a function of time.

(a) What is the amplitude of the oscillations?
(b) What is the period of the oscillations?
(c) What is the average value of the voltage?
(d) What is the shift along the $t$ axis, $t_0$?
(e) What is a formula for this function?
Activity 0.18.

Suppose the following sinusoidal function models the water level on a pier in the ocean as it changes due to the tides during a certain day.

\[ w(t) = 4.3 \sin (0.51t + 0.82) + 10.6 \]

(a) Using the formula above, make a table showing the water level every two hours for a 24 hour period starting at midnight.

<table>
<thead>
<tr>
<th>time (hours)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>22</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>water level (ft)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Sketch a graph of this function using the data from your table in part (a).

(c) What is the period of oscillation of this function?

(d) What time is high tide?
Voting Questions

0.5.1 Which of the following is the approximate value for the sine and cosine of angles \( A \) and \( B \) in the figure below.

(a) \( \sin A \approx 0.5, \cos A \approx 0.85, \sin B \approx -0.7, \cos B \approx 0.7 \)
(b) \( \sin A \approx 0.85, \cos A \approx 0.5, \sin B \approx -0.7, \cos B \approx 0.7 \)
(c) \( \sin A \approx 0.5, \cos A \approx 0.85, \sin B \approx 0.7, \cos B \approx 0.7 \)
(d) \( \sin A \approx 0.85, \cos A \approx 0.5, \sin B \approx 0.7, \cos B \approx 0.7 \)

0.5.2 The amplitude and period of the function below are

(a) Amplitude = 2, Period = 2
(b) Amplitude = 2, Period = 3
(c) Amplitude = 2, Period = 1/2
(d) Amplitude = 3, Period = 2
(e) Amplitude = 3, Period = 1/2

0.5.3 What is the equation of the function shown in the graph?
0.5. TRIGONOMETRIC FUNCTIONS

(a) \( y = 3 \sin(2x) + 2 \)
(b) \( y = 3 \cos(2x) + 2 \)
(c) \( y = 3 \sin(\pi x) + 2 \)
(d) \( y = 3 \cos(\pi x) + 2 \)
(e) \( y = 3 \sin\left(\frac{1}{3}x\right) + 2 \)
(f) \( y = 3 \cos\left(\frac{1}{3}x\right) + 2 \)

0.5.4 The amplitude and period of the function below are

(a) Amplitude = 2, Period = 2
(b) Amplitude = 2, Period = 3
(c) Amplitude = 2, Period = 1/2
(d) Amplitude = 3, Period = 2
(e) Amplitude = 3, Period = 1/2

0.5.5 Which of the following could describe the graph below?

(a) \( y = 3 \cos(2x) \)
(b) \( y = 3 \cos(x/2) \)
(c) \( y = 3 \sin(2x) \)
(d) \( y = 3 \sin(x/2) \)
0.5.6 The function \( f(x) = 3 \sin(2x + 4) \) is created when you take the function \( g(x) = 3 \sin(2x) \) and you...

(a) shift it left by 4 units.
(b) shift it right by 4 units.
(c) shift it left by 2 units.
(d) shift it right by 2 units.
(e) shift it left by 8 units.

0.5.7 Which of the following could describe the graph below?

(a) \( y = 4 \sin \left( \pi x - \frac{\pi}{2} \right) - 2 \)
(b) \( y = -4 \sin \left( \pi x + \frac{\pi}{2} \right) - 2 \)
(c) \( y = -4 \cos(\pi x) - 2 \)
(d) \( y = 4 \cos(\pi(x + 1)) - 2 \)
(e) All of the above
(f) More than one, but not all of the above

0.5.8 What is an equation of the function whose graph is given below?

(a) \( f(x) = \cot x \)
(b) \( f(x) = \cot 2x \)
(c) \( f(x) = \cot \left( x - \frac{\pi}{2} \right) \)
(d) \( f(x) = \cot \left( 2x - \frac{\pi}{2} \right) \)
0.5.9 Three different functions of the form \( y = A \sin(Bx + C) \) are plotted below. Could these all have the same value of \( B \)?

(a) Yes
(b) No
(c) Not enough information is given.

0.5.10 The functions plotted below are all of the form \( y = A \sin(Bx + C) \). Which function has the largest value of \( B \)?
0.5.11 What is the phase shift of \( f(x) = \frac{1}{5} \tan \left( 2x + \frac{\pi}{2} \right) \)?

(a) \( 2\pi \)
(b) \( \pi \)
(c) \( \frac{\pi}{2} \)
(d) \( \frac{\pi}{4} \)
(e) \( -2\pi \)
(f) \( -\pi \)
(g) \( -\frac{\pi}{2} \)
(h) \( -\frac{\pi}{4} \)

0.5.12 What is the amplitude of \( f(x) = -3 \sin(2x) \)?

(a) 3
(b) -3
(c) \( \pi \)
(d) \( 2\pi \)

0.5.13 What is the amplitude of \( f(x) = -2 \sin x \)?

(a) 1
0.5.14 What is the period of $f(x) = -3 \sin(2x)$?

(a) 3
(b) -3
(c) $\pi$
(d) $2\pi$

0.5.15 What is the period of $f(x) = \frac{1}{5} \tan(2x)$?

(a) $\frac{1}{5}$
(b) $2\pi$
(c) $\pi$
(d) $\frac{\pi}{2}$
(e) $\frac{\pi}{4}$

0.5.16 Which of the basic trig functions below are odd functions?

(a) $f(x) = \sin(x)$.
(b) $f(x) = \cos(x)$.
(c) $f(x) = \tan(x)$.
(d) (a) and (b).
(e) (a) and (c).
(f) (b) and (c).
(g) (a), (b), and (c).
(h) None of the above.
0.6 Powers, Polynomials, and Rational Functions

Preview Activity 0.6. Figure 4 shows the graphs of two different functions. Suppose that you were to graph a line anywhere along each of the two graphs.

1. Is it possible to draw a line that does not intersect the graph of \( f \)? \( g \)?
2. Is it possible to draw a line that intersects the graph of \( f \) an even number of times?
3. Is it possible to draw a line that intersects the graph of \( g \) an odd number of times?
4. What is the fewest number of intersections that your line could have with the graph of \( f \) with \( g \)?
5. What is the largest number of intersections that your line could have with the graph of \( f \) with \( g \)?
6. How many times does the graph of \( f \) change directions? How many times does the graph of \( g \) change directions?

Figure 4: \( f(x) \) and \( g(x) \) for the preview activity.
Activity 0.19.

Power functions and exponential functions appear somewhat similar in their formulas, but behave differently in many ways.

(a) Compare the functions $f(x) = x^2$ and $g(x) = 2^x$ by graphing both functions in several viewing windows. Find the points of intersection of the graphs. Which function grows more rapidly when $x$ is large?

(b) Compare the functions $f(x) = x^{10}$ and $g(x) = 2^x$ by graphing both functions in several viewing windows. Find the points of intersection of the graphs. Which function grows more rapidly when $x$ is large?

(c) Make a conjecture: As $x \rightarrow \infty$, which dominates, $x^n$ or $a^x$ for $a > 1$?
Activity 0.20.

For each of the following graphs, find a possible formula for the polynomial of lowest degree that fits the graph.
Activity 0.21.

(a) Suppose $f(x) = x^2 + 3x + 2$ and $g(x) = x - 3$.

(i) What is the behavior of the function $h(x) = \frac{f(x)}{g(x)}$ near $x = -1$? (i.e. what happens to $h(x)$ as $x \to -1$?) near $x = -2$? near $x = 3$?

(ii) What is the behavior of the function $k(x) = \frac{g(x)}{f(x)}$ near $x = -1$? near $x = -2$? near $x = 3$?

(b) Suppose $f(x) = x^2 - 9$ and $g(x) = x - 3$.

(i) What is the behavior of the function $h(x) = \frac{f(x)}{g(x)}$ near $x = -3$? (i.e. what happens to $h(x)$ as $x \to -3$?) near $x = 3$?

(ii) What is the behavior of the function $k(x) = \frac{g(x)}{f(x)}$ near $x = -3$? near $x = 3$?
Activity 0.22.

(a) Suppose $f(x) = x^3 + 2x^2 - x + 7$ and $g(x) = x^2 + 4x + 2$.
   (i) Which function dominates as $x \to \infty$?
   (ii) What is the behavior of the function $h(x) = \frac{f(x)}{g(x)}$ as $x \to \infty$?
   (iii) What is the behavior of the function $k(x) = \frac{g(x)}{f(x)}$ as $x \to \infty$?

(b) Suppose $f(x) = 2x^4 - 5x^3 + 8x^2 - 3x - 1$ and $g(x) = 3x^4 - 2x^2 + 1$
   (i) Which function dominates as $x \to \infty$?
   (ii) What is the behavior of the function $h(x) = \frac{f(x)}{g(x)}$ as $x \to \infty$?
   (iii) What is the behavior of the function $k(x) = \frac{g(x)}{f(x)}$ as $x \to \infty$?

(c) Suppose $f(x) = e^x$ and $g(x) = x^{10}$.
   (i) Which function dominates as $x \to \infty$ as $x \to \infty$?
   (ii) What is the behavior of the function $h(x) = \frac{f(x)}{g(x)}$ as $x \to \infty$?
   (iii) What is the behavior of the function $k(x) = \frac{g(x)}{f(x)}$ as $x \to \infty$?
Activity 0.23.

For each of the following functions, determine (1) whether the function has a horizontal asymptote, and (2) whether the function crosses its horizontal asymptote.

(a) \( f(x) = \frac{x + 3}{5x - 2} \)

(b) \( g(x) = \frac{x^2 + 2x - 1}{x - 1} \)

(c) \( h(x) = \frac{x + 1}{x^2 + 2x - 1} \)
Voting Questions

0.6.1 Which of the following is not a power function?

(a) \( f(x) = 3x^2 \)
(b) \( f(x) = x^{1.5} \)
(c) \( f(x) = 6 \cdot 2^x \)
(d) \( f(x) = -3x^{-π} \)

0.6.2 As \( x \to \infty \), which function dominates? That is, which function is larger in the long run?

(a) \( 0.1x^2 \)
(b) \( 10^{10}x \)

0.6.3 As \( x \to \infty \), which function dominates?

(a) \( 0.25\sqrt{x} \)
(b) \( 25,000x^{-3} \)

0.6.4 As \( x \to \infty \), which function dominates?

(a) \( 3 - 0.9^x \)
(b) \( \log x \)

0.6.5 Which function dominates as \( x \to \infty \)?

(a) \( x^2 \)
(b) \( e^x \)

0.6.6 As \( x \to \infty \), which function dominates?

(a) \( x^3 \)
(b) \( 2^x \)

0.6.7 As \( x \to \infty \), which function dominates?
0.6.8 Which of these functions dominates as $x \to \infty$?

(a) $f(x) = -5x$
(b) $g(x) = 10^x$
(c) $h(x) = 0.9^x$
(d) $k(x) = x^5$
(e) $l(x) = \pi^x$

0.6.9 If $f(x) = ax^2 + bx + c$ is a quadratic function, then the lowest point on the graph of $f(x)$ occurs at $x = -b/2a$. 

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.6.10 Under what condition is the graph of the quadratic function described by $f(x) = ax^2 + bx + c$ concave down?

(a) $a < 0$.
(b) $b < 0$.
(c) $c < 0$.
(d) More than one of the above.
(e) None of the above.

0.6.11 What is the degree of the graph of the polynomial in the figure below?
0.6.12 Which of the options below describes a function which is even?

(a) Any polynomial of even degree.
(b) Any polynomial of odd degree.
(c) \( f(x) = 9x^6 - 3x^2 + 2. \)
(d) \( f(x) = 3x^4 - 2x^3 + x^2. \)
(e) More than 1 of the above.
(f) None of the above.

0.6.13 The equation \( y = x^3 + 2x^2 - 5x - 6 \) is represented by which graph?
0.6.14 The graph below is a representation of which function?

(a) $y = 3x + 2$
(b) $y = (x - 2)(x + 3)$
(c) $y = (x - 6)(x - 2)$
(d) $y = (x - 3)(x + 2)$
(e) none of these

0.6.15 Let $f(x) = \frac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$, then $f(x) = g(x)$.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

0.6.16 Which if the following is a graph for $y = \frac{1-x^2}{x+2}$. (No calculators allowed.)
0.6.17 Which of the graphs represents \( y = \frac{2x}{x-2} \)?
Chapter 1

Understanding the Derivative

1.1 How do we measure velocity?

Preview Activity 1.1. This is the first Preview Activity in this text. Your job for this activity is to get to know the textbook.

(a) Where can you find the full textbook?

(b) What chapters of this text are you going to cover this semester. Have a look at your syllabus!

(c) What are the differences between Preview Activities, Activities, Examples, Exercises, Voting Questions, and WeBWork? Which ones should you do before class, which ones will you likely do during class, and which ones should you be doing after class?

(d) What materials in this text would you use to prepare for an exam and where do you find them?

(e) What should you bring to class every day?
Preview Activity 1.2. Suppose that the height \( s \) of a ball (in feet) at time \( t \) (in seconds) is given by the formula \( s(t) = 64 - 16(t - 1)^2 \).

(a) Construct an accurate graph of \( y = s(t) \) on the time interval \( 0 \leq t \leq 3 \). Label at least six distinct points on the graph, including the three points that correspond to when the ball was released, when the ball reaches its highest point, and when the ball lands.

(b) In everyday language, describe the behavior of the ball on the time interval \( 0 < t < 1 \) and on time interval \( 1 < t < 3 \). What occurs at the instant \( t = 1 \)?

(c) Consider the expression

\[
AV_{[0.5,1]} = \frac{s(1) - s(0.5)}{1 - 0.5}.
\]

Compute the value of \( AV_{[0.5,1]} \). What does this value measure geometrically? What does this value measure physically? In particular, what are the units on \( AV_{[0.5,1]} \)?
Activity 1.1.

The following questions concern the position function given by \( s(t) = 64 - 16(t - 1)^2 \), which is the same function considered in Preview Activity 1.2.

(a) Compute the average velocity of the ball on each of the following time intervals: 
   \([0.4, 0.8], [0.7, 0.8], [0.79, 0.8], [0.799, 0.8], [0.8, 1.2], [0.8, 0.9], [0.8, 0.81], [0.8, 0.801]\). Include units for each value.

(b) On the provided graph in Figure 1.1, sketch the line that passes through the points \( A = (0.4, s(0.4)) \) and \( B = (0.8, s(0.8)) \). What is the meaning of the slope of this line? In light of this meaning, what is a geometric way to interpret each of the values computed in the preceding question?

(c) Use a graphing utility to plot the graph of \( s(t) = 64 - 16(t - 1)^2 \) on an interval containing the value \( t = 0.8 \). Then, zoom in repeatedly on the point \( (0.8, s(0.8)) \). What do you observe about how the graph appears as you view it more and more closely?

(d) What do you conjecture is the velocity of the ball at the instant \( t = 0.8 \)? Why?

![Figure 1.1: A partial plot of \( s(t) = 64 - 16(t - 1)^2 \).](image-url)
Activity 1.2.

Each of the following questions concern $s(t) = 64 - 16(t - 1)^2$, the position function from Preview Activity 1.2.

(a) Compute the average velocity of the ball on the time interval $[1.5, 2]$. What is different between this value and the average velocity on the interval $[0, 0.5]$?

(b) Use appropriate computing technology to estimate the instantaneous velocity of the ball at $t = 1.5$. Likewise, estimate the instantaneous velocity of the ball at $t = 2$. Which value is greater?

(c) How is the sign of the instantaneous velocity of the ball related to its behavior at a given point in time? That is, what does positive instantaneous velocity tell you the ball is doing? Negative instantaneous velocity?

(d) Without doing any computations, what do you expect to be the instantaneous velocity of the ball at $t = 1$? Why?
Activity 1.3.

For the function given by \( s(t) = 64 - 16(t - 1)^2 \) from Preview Activity 1.2, find the most simplified expression you can for the average velocity of the ball on the interval \([2, 2 + h]\). Use your result to compute the average velocity on \([1.5, 2]\) and to estimate the instantaneous velocity at \( t = 2 \). Finally, compare your earlier work in Activity 1.1.
Voting Questions

1.1.1 The speedometer in my car is broken. In order to find my average velocity on a trip from Helena to Missoula, I need

i. the distance between Helena and Missoula
ii. the time spent traveling
iii. the number of stops I made during the trip
iv. a friend with a stopwatch
v. a working odometer
vi. none of the above

Select the best combination:

(a) i, ii, & iii only
(b) i & ii only
(c) iv & v only
(d) vi
(e) a combination that is not listed here

1.1.2 The speedometer in my car is broken. In order to find my velocity at the instant I hit a speed trap, I need

i. the distance between Helena and Missoula
ii. the time spent traveling
iii. the number of stops I made during the trip
iv. a friend with a stopwatch
v. a working odometer
vi. none of the above

Select the best combination:

(a) i, ii, & iii only
(b) i & ii only
(c) iv & v only
(d) vi
(e) a combination that is not listed here
1.1.3 Which graph represents an object slowing down, where $D$ is distance, and $t$ is time? Assume that the units are the same for all graphs.

1.1.4 True or False: If a car is going 50 miles per hour at 2 pm and 60 miles per hour at 3 pm, then it travels between 50 and 60 miles during the hour between 2 pm and 3 pm.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.1.5 True or False: If a car travels 80 miles between 2 and 4 pm, then its velocity is close to 40 mph at 2 pm.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.1.6 True or False: If the time interval is short enough, then the average velocity of a car over the time interval and the instantaneous velocity at a time in the interval can be expected to be close.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
1.1.7 True or False: If an object moves with the same average velocity over every time interval, then its average velocity equals its instantaneous velocity at any time.

(a) True, and I am very confident

(b) True, but I am not very confident

(c) False, but I am not very confident

(d) False, and I am very confident
1.2 The notion of limit

Preview Activity 1.3. Suppose that $g$ is the function given by the graph below. Use the graph to answer each of the following questions.

(a) Determine the values $g(-2)$, $g(-1)$, $g(0)$, $g(1)$, and $g(2)$, if defined. If the function value is not defined, explain what feature of the graph tells you this.

(b) For each of the values $a = -1$, $a = 0$, and $a = 2$, complete the following sentence: “As $x$ gets closer and closer (but not equal) to $a$, $g(x)$ gets as close as we want to ____.”

(c) What happens as $x$ gets closer and closer (but not equal) to $a = 1$? Does the function $g(x)$ get as close as we would like to a single value?

![Graph of $y = g(x)$ for Preview Activity 1.3.](image)
Activity 1.4.

Estimate the value of each of the following limits by constructing appropriate tables of values. Then determine the exact value of the limit by using algebra to simplify the function. Finally, plot each function on an appropriate interval to check your result visually.

(a) \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \)

(b) \( \lim_{x \to 0} \frac{(2 + x)^3 - 8}{x} \)

(c) \( \lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} \)
Activity 1.5.

Consider a moving object whose position function is given by \( s(t) = t^2 \), where \( s \) is measured in meters and \( t \) is measured in minutes.

(a) Determine a simplified expression for the average velocity of the object on the interval \([3, 3 + h]\).

(b) Determine the average velocity of the object on the interval \([3, 3.2]\). Include units on your answer.

(c) Determine the instantaneous velocity of the object when \( t = 3 \). Include units on your answer.
Activity 1.6.

For the moving object whose position $s$ at time $t$ is given by the graph below, answer each of the following questions. Assume that $s$ is measured in feet and $t$ is measured in seconds.

![Graph of position function $s(t)$](image.png)

Figure 1.3: Plot of the position function $y = s(t)$ in Activity 1.6.

(a) Use the graph to estimate the average velocity of the object on each of the following intervals: $[0.5, 1]$, $[1.5, 2.5]$, $[0, 5]$. Draw each line whose slope represents the average velocity you seek.

(b) How could you use average velocities and slopes of lines to estimate the instantaneous velocity of the object at a fixed time?

(c) Use the graph to estimate the instantaneous velocity of the object when $t = 2$. Should this instantaneous velocity at $t = 2$ be greater or less than the average velocity on $[1.5, 2.5]$ that you computed in (a)? Why?
1.2. THE NOTION OF LIMIT

Voting Questions

1.2.1 Consider the function:

\[
f(x) = \begin{cases} 
6 & \text{if } x > 9 \\
2 & \text{if } x = 9 \\
-x + 14 & \text{if } -7 \leq x < 9 \\
21 & \text{if } x < -7 
\end{cases}
\]

(a) \( \lim_{x \to 9^-} f(x) = 2 \)
(b) \( \lim_{x \to 9^-} f(x) = 5 \)
(c) \( \lim_{x \to 9^-} f(x) = 6 \)
(d) \( \lim_{x \to 9^-} f(x) = 14 \)
(e) \( \lim_{x \to 9^-} f(x) = 21 \)

1.2.2 True or False: As \( x \) increases to 100, \( f(x) = 1/x \) gets closer and closer to 0, so the limit as \( x \) goes to 100 of \( f(x) \) is 0. Be prepared to justify your answer.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.2.3 True or False: \( \lim_{x \to a} f(x) = L \) means that if \( x_1 \) is closer to \( a \) than \( x_2 \) is, then \( f(x_1) \) will be closer to \( L \) than \( f(x_2) \) is. Be prepared to justify your answer with an argument or counterexample.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.2.4 The reason that \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) does not exist is:

(a) because no matter how close \( x \) gets to 0, there are \( x \)'s near 0 for which \( \sin \left( \frac{1}{x} \right) = 1 \), and some for which \( \sin \left( \frac{1}{x} \right) = -1 \).

(b) because the function values oscillate around 0.
(c) because $\frac{1}{0}$ is undefined.
(d) all of the above

1.2.5 $\lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right)$

(a) does not exist because no matter how close $x$ gets to 0, there are $x$’s near 0 for which $\sin \left( \frac{1}{x} \right) = 1$, and some for which $\sin \left( \frac{1}{x} \right) = -1$.
(b) does not exist because the function values oscillate around 0.
(c) does not exist because $\frac{1}{0}$ is undefined.
(d) equals 0
(e) equals 1

1.2.6 You’re trying to guess $\lim_{x \to 0} f(x)$. You plug in $x = 0.1, 0.01, 0.001, \ldots$ and get $f(x) = 0$ for all of these values. In fact you’re told that for all $n = 1, 2, \ldots, f \left( \frac{1}{10^n} \right) = 0$. True or False: Since the sequence $f(0.1), f(0.01), f(0.001), \ldots$ goes to 0, we know that $\lim_{x \to 0} = 0$.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.2.7 If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$

(a) does not exist.
(b) must exist.
(c) can’t be determined. Not enough information is given.

1.2.8 True or False: Consider a function $f(x)$ with the property that $\lim_{x \to a} f(x) = 0$. Now consider another function $g(x)$ also defined near $a$. Then $\lim_{x \to a} [f(x)g(x)] = 0$.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
1.2.9 If a function \( f \) is not defined at \( x = a \),

(a) \( \lim_{x \to a} \) cannot exist.
(b) \( \lim_{x \to a} \) could be 0.
(c) \( \lim_{x \to a} \) must approach \( \infty \).
(d) none of the above

1.2.10 Possible criteria for continuity at a point: If the limit of the function exists at a point, the function is continuous at that point. Which of the following examples fits the above criteria but is not continuous at \( x = 0 \)?

(a) \( f(x) = x \)
(b) \( f(x) = \frac{x^2}{x} \)
(c) \( f(x) = \frac{|x|}{x} \)
(d) None of these show a problem with this criteria.

1.2.11 Let \( f(x) = 5x^4 + 18x^3 - 2x + 3 \). As \( x \) gets really big, what becomes the most important (dominant) term in this function?

(a) \( 5x^4 \)
(b) \( 18x^3 \)
(c) \( -2x \)
(d) \( 3 \)

1.2.12 What is

\[
\lim_{x \to \infty} \frac{6x^2 - 5x}{2x^2 + 3}?
\]

(a) 0
(b) 2
(c) 3
(d) 6
(e) infinity

1.2.13 What is

\[
\lim_{x \to \infty} \frac{3x^2 + 5x^3 - 2x + 4}{4x^3 - 5x + 6}?
\]
1.2. THE NOTION OF LIMIT

(a) 0
(b) 2/3
(c) 3/4
(d) 5/4
(e) infinity

1.2.14 What is \( \lim_{x \to \infty} \frac{100x^5 - 15x}{x^6 + 3} \) ?

(a) 0
(b) 5/6
(c) 85
(d) 100
(e) infinity

1.2.15 What is \( \lim_{x \to \infty} \frac{x^2 + 2x + 3}{25x - 7} \) ?

(a) 0
(b) 1/25
(c) 3/7
(d) 2
(e) infinity

1.2.16 Let \( f(x) = \frac{x^2 - 4x + 3}{x^2 - 1} \). Evaluate \( \lim_{x \to -1^+} f(x) \).

(a) \(-1\)
(b) \(\infty\)
(c) \(-\infty\)
1.3 The derivative of a function at a point

Preview Activity 1.4. Suppose that \( f \) is the function given by the graph below and that \( a \) and \( a+h \) are the input values as labeled on the \( x \)-axis. Use the graph in Figure 1.4 to answer the following questions.

![Graph of \( y = f(x) \) for Preview Activity 1.4.](image)

(a) Locate and label the points \((a, f(a))\) and \((a+h, f(a+h))\) on the graph.

(b) Construct a right triangle whose hypotenuse is the line segment from \((a, f(a))\) to \((a+h, f(a+h))\). What are the lengths of the respective legs of this triangle?

(c) What is the slope of the line that connects the points \((a, f(a))\) and \((a+h, f(a+h))\)?

(d) Write a meaningful sentence that explains how the average rate of change of the function on a given interval and the slope of a related line are connected.
Activity 1.7.

Consider the function $f$ whose formula is $f(x) = 3 - 2x$.

(a) What familiar type of function is $f$? What can you say about the slope of $f$ at every value of $x$?

(b) Compute the average rate of change of $f$ on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5+h]$; simplify each result as much as possible. What do you notice about these quantities?

(c) Use the limit definition of the derivative to compute the exact instantaneous rate of change of $f$ with respect to $x$ at the value $a = 1$. That is, compute $f'(1)$ using the limit definition. Show your work. Is your result surprising?

(d) Without doing any additional computations, what are the values of $f'(2)$, $f'(\pi)$, and $f'(-\sqrt{2})$? Why?
Activity 1.8.

A water balloon is tossed vertically in the air from a window. The balloon’s height in feet at time \( t \) in seconds after being launched is given by \( s(t) = -16t^2 + 16t + 32 \). Use this function to respond to each of the following questions.

(a) Sketch an accurate, labeled graph of \( s \) on the axes provided in Figure 1.5. You should be able to do this without using computing technology.

(b) Compute the average rate of change of \( s \) on the time interval \([1, 2]\). Include units on your answer and write one sentence to explain the meaning of the value you found.

(c) Use the limit definition to compute the instantaneous rate of change of \( s \) with respect to time, \( t \), at the instant \( a = 1 \). Show your work using proper notation, include units on your answer, and write one sentence to explain the meaning of the value you found.

(d) On your graph in (a), sketch two lines: one whose slope represents the average rate of change of \( s \) on \([1, 2]\), the other whose slope represents the instantaneous rate of change of \( s \) at the instant \( a = 1 \). Label each line clearly.

(e) For what values of \( a \) do you expect \( s'(a) \) to be positive? Why? Answer the same questions when “positive” is replaced by “negative” and “zero.”
Activity 1.9.

A rapidly growing city in Arizona has its population \( P \) at time \( t \), where \( t \) is the number of decades after the year 2010, modeled by the formula \( P(t) = 25000e^{t/5} \). Use this function to respond to the following questions.

(a) Sketch an accurate graph of \( P \) for \( t = 0 \) to \( t = 5 \) on the axes provided in Figure 1.6. Label the scale on the axes carefully.

(b) Compute the average rate of change of \( P \) between 2030 and 2050. Include units on your answer and write one sentence to explain the meaning (in everyday language) of the value you found.

(c) Use the limit definition to write an expression for the instantaneous rate of change of \( P \) with respect to time, \( t \), at the instant \( a = 2 \). Explain why this limit is difficult to evaluate exactly.

(d) Estimate the limit in (c) for the instantaneous rate of change of \( P \) at the instant \( a = 2 \) by using several small \( h \) values. Once you have determined an accurate estimate of \( P'(2) \), include units on your answer, and write one sentence (using everyday language) to explain the meaning of the value you found.

(e) On your graph above, sketch two lines: one whose slope represents the average rate of change of \( P \) on \([2, 4]\), the other whose slope represents the instantaneous rate of change of \( P \) at the instant \( a = 2 \).

(f) In a carefully-worded sentence, describe the behavior of \( P'(a) \) as \( a \) increases in value. What does this reflect about the behavior of the given function \( P \)?
Voting Questions

1.3.1 We want to find how the volume of a balloon changes as it is filled with air. We know \( V(r) = \frac{4}{3} \pi r^3 \), where \( r \) is the radius in inches and \( V(r) \) is the volume in cubic inches. The expression \( \frac{V(3) - V(1)}{3 - 1} \) represents the

(a) Average rate of change of the radius with respect to the volume when the radius changes from 1 inch to 3 inches.
(b) Average rate of change of the radius with respect to the volume when the volume changes from 1 cubic inch to 3 cubic inches.
(c) Average rate of change of the volume with respect to the radius when the radius changes from 1 inch to 3 inches.
(d) Average rate of change of the volume with respect to the radius when the volume changes from 1 cubic inch to 3 cubic inches.

1.3.2 We want to find how the volume of a balloon changes as it is filled with air. We know \( V(r) = \frac{4}{3} \pi r^3 \), where \( r \) is the radius in inches and \( V(r) \) is the volume in cubic inches. Which of the following represents the rate at which the volume is changing when the radius is 1 inch?

(a) \( \frac{V(1.01) - V(1)}{0.01} \approx 12.69 \)
(b) \( \frac{V(0.99) - V(1)}{-0.01} \approx 12.44 \)
(c) \( \lim_{h \to 0} \frac{V(1+h) - V(1)}{h} \)
(d) All of the above

1.3.3 Which of the following represents the slope of a line drawn between the two points marked in the figure?

(a) \( m = \frac{F(a)+F(b)}{a+b} \)
(b) \( m = \frac{F(b)-F(a)}{b-a} \)
(c) \( m = \frac{a}{b} \)
(d) \( m = \frac{F(a)-F(b)}{b-a} \)
1.3.4 The line tangent to the graph of \( f(x) = x \) at \((0,0)\)

(a) is \( y = 0 \)
(b) is \( y = x \)
(c) does not exist
(d) is not unique. There are infinitely many tangent lines.

1.3.5 Suppose that \( f(x) \) is a function with \( f(2) = 15 \) and \( f'(2) = 3 \). Estimate \( f(2.5) \).

(a) 10.5
(b) 15
(c) 16.5
(d) 18
1.4 The derivative function

Preview Activity 1.5. Consider the function $f(x) = 4x - x^2$.

(a) Use the limit definition to compute the following derivative values: $f'(0)$, $f'(1)$, $f'(2)$, and $f'(3)$.

(b) Observe that the work to find $f'(a)$ is the same, regardless of the value of $a$. Based on your work in (a), what do you conjecture is the value of $f'(4)$? How about $f'(5)$? (Note: you should not use the limit definition of the derivative to find either value.)

(c) Conjecture a formula for $f'(a)$ that depends only on the value $a$. That is, in the same way that we have a formula for $f(x)$ (recall $f(x) = 4x - x^2$), see if you can use your work above to guess a formula for $f'(a)$ in terms of $a$. 

\[\boxed{\square}\]
Activity 1.10.

For each given graph of \( y = f(x) \), sketch an approximate graph of its derivative function, \( y = f'(x) \), on the axes immediately below. The scale of the grid for the graph of \( f \) is \( 1 \times 1 \); assume the horizontal scale of the grid for the graph of \( f' \) is identical to that for \( f \). If necessary, adjust and label the vertical scale on the axes for the graph of \( f' \).
Write several sentences that describe your overall process for sketching the graph of the derivative function, given the graph the original function. What are the values of the derivative function that you tend to identify first? What do you do thereafter? How do key traits of the graph of the derivative function exemplify properties of the graph of the original function?
Activity 1.11.

For each of the listed functions, determine a formula for the derivative function. For the first two, determine the formula for the derivative by thinking about the nature of the given function and its slope at various points; do not use the limit definition. For the latter four, use the limit definition. Pay careful attention to the function names and independent variables. It is important to be comfortable with using letters other than $f$ and $x$. For example, given a function $p(z)$, we call its derivative $p'(z)$.

(a) $f(x) = 1$
(b) $g(t) = t$
(c) $p(z) = z^2$
(d) $q(s) = s^3$
(e) $F(t) = \frac{1}{t}$
(f) $G(y) = \sqrt{y}$
1.4.1 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.6?

1.4.2 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.8?

1.4.3 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.9?
1.4.4 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.10?

1.4.5 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.11?
1.4.6 The graph in Figure 2.12 is the derivative of which of the following functions?

1.4.7 The graph in Figure 2.13 is the derivative of which of the following functions?

1.4.8 The graph in Figure 2.14 is the derivative of which of the following functions?
1.4.9 The graph in Figure 2.15 is the derivative of which of the following functions?

1.4.10 True or False: If \( f'(x) = g'(x) \) then \( f(x) = g(x) \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.4.11 Let \( f(x) = 2x^3 + 3x^2 + 1 \). True or false: On the interval \( (-\infty, -1) \), the function \( f \) is increasing,

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.
1.5 Interpreting the derivative and its units

Preview Activity 1.6. One of the longest stretches of straight (and flat) road in North America can be found on the Great Plains in the state of North Dakota on state highway 46, which lies just south of the interstate highway I-94 and runs through the town of Gackle. A car leaves town (at time $t = 0$) and heads east on highway 46; its position in miles from Gackle at time $t$ in minutes is given by the graph of the function in Figure 1.7. Three important points are labeled on the graph; where the curve looks linear, assume that it is indeed a straight line.

![Figure 1.7: The graph of $y = s(t)$, the position of the car along highway 46, which tells its distance in miles from Gackle, ND, at time $t$ in minutes.](image)

(a) In everyday language, describe the behavior of the car over the provided time interval. In particular, discuss what is happening on the time intervals $[57, 68]$ and $[68, 104]$.

(b) Find the slope of the line between the points $(57, 63.8)$ and $(104, 106.8)$. What are the units on this slope? What does the slope represent?

(c) Find the average rate of change of the car’s position on the interval $[68, 104]$. Include units on your answer.

(d) Estimate the instantaneous rate of change of the car’s position at the moment $t = 80$. Write a sentence to explain your reasoning and the meaning of this value.
Activity 1.12.

A potato is placed in an oven, and the potato’s temperature $F$ (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time $t$ is measured in minutes.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$F(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>70</td>
</tr>
<tr>
<td>15</td>
<td>180.5</td>
</tr>
<tr>
<td>30</td>
<td>251</td>
</tr>
<tr>
<td>45</td>
<td>296</td>
</tr>
<tr>
<td>60</td>
<td>324.5</td>
</tr>
<tr>
<td>75</td>
<td>342.8</td>
</tr>
<tr>
<td>90</td>
<td>354.5</td>
</tr>
</tbody>
</table>

(a) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t = 30$. Include units on your answer.

(b) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t = 60$. Include units on your answer.

(c) Without doing any calculation, which do you expect to be greater: $F'(75)$ or $F'(90)$? Why?

(d) Suppose it is given that $F(64) = 330.28$ and $F'(64) = 1.341$. What are the units on these two quantities? What do you expect the temperature of the potato to be when $t = 65$? When $t = 66$? Why?

(e) Write a couple of careful sentences that describe the behavior of the temperature of the potato on the time interval $[0, 90]$, as well as the behavior of the instantaneous rate of change of the temperature of the potato on the same time interval.
Activity 1.13.

A company manufactures rope, and the total cost of producing \( r \) feet of rope is \( C(r) \) dollars.

(a) What does it mean to say that \( C(2000) = 800 \)?

(b) What are the units of \( C'(r) \)?

(c) Suppose that \( C(2000) = 800 \) and \( C'(2000) = 0.35 \). Estimate \( C(2100) \), and justify your estimate by writing at least one sentence that explains your thinking.

(d) Which of the following statements do you think is true, and why?

- \( C'(2000) < C'(3000) \)
- \( C'(2000) = C'(3000) \)
- \( C'(2000) > C'(3000) \)

(e) Suppose someone claims that \( C'(5000) = -0.1 \). What would the practical meaning of this derivative value tell you about the approximate cost of the next foot of rope? Is this possible? Why or why not?
Activity 1.14.

Researchers at a major car company have found a function that relates gasoline consumption to speed for a particular model of car. In particular, they have determined that the consumption $C$, in liters per kilometer, at a given speed $s$, is given by a function $C = f(s)$, where $s$ is the car’s speed in kilometers per hour.

(a) Data provided by the car company tells us that $f(80) = 0.015$, $f(90) = 0.02$, and $f(100) = 0.027$. Use this information to estimate the instantaneous rate of change of fuel consumption with respect to speed at $s = 90$. Be as accurate as possible, use proper notation, and include units on your answer.

(b) By writing a complete sentence, interpret the meaning (in the context of fuel consumption) of “$f(80) = 0.015$.”

(c) Write at least one complete sentence that interprets the meaning of the value of $f'(90)$ that you estimated in (a).
1.5. INTERPRETING THE DERIVATIVE AND ITS UNITS

Voting Questions

1.5.1 The radius of a snowball changes as the snow melts. The instantaneous rate of change in radius with respect to volume is

(a) \( \frac{dV}{dr} \)
(b) \( \frac{dr}{dV} \)
(c) \( \frac{dV}{dr} + \frac{dr}{dV} \)
(d) None of the above

1.5.2 Gravel is poured into a conical pile. The rate at which gravel is added to the pile is

(a) \( \frac{dV}{dt} \)
(b) \( \frac{dr}{dt} \)
(c) \( \frac{dV}{dr} \)
(d) None of the above

1.5.3 A slow freight train chugs along a straight track. The distance it has traveled after \( x \) hours is given by a function \( f(x) \). An engineer is walking along the top of the box cars at the rate of 3 mi/hr in the same direction as the train is moving. The speed of the man relative to the ground is

(a) \( f(x) + 3 \)
(b) \( f'(x) + 3 \)
(c) \( f(x) - 3 \)
(d) \( f'(x) - 3 \)

1.5.4 \( C(r) \) gives the total cost of paying off a car loan that has an annual interest rate of \( r \) %. What are the units of \( C'(r) \)?

(a) Year / $
(b) $ / Year
(c) $ / %
(d) % / $
1.5.5 \( C(r) \) gives the total cost of paying off a car loan that has an annual interest rate of \( r \% \). What is the practical meaning of \( C'(5) \)?

(a) The rate of change of the total cost of the car loan is \( C'(5) \).
(b) If the interest rate increases by 1\%, then the total cost of the loan increases by \( C'(5) \).
(c) If the interest rate increases by 1\%, then the total cost of the loan increases by \( C'(5) \) when the interest rate is 5\%.
(d) If the interest rate increases by 5\%, then the total cost of the loan increases by \( C'(5) \).

1.5.6 \( C(r) \) gives the total cost of paying off a car loan that has an annual interest rate of \( r \% \). What is the sign of \( C'(5) \)?

(a) Positive
(b) Negative
(c) Not enough information is given

1.5.7 \( g(v) \) gives the fuel efficiency, in miles per gallon, of a car going a speed of \( v \) miles per hour. What are the units of \( g'(v) = \frac{dg}{dv} \)?

(a) (miles)²/[(gal)(hour)]
(b) hour/gal
(c) gal/hour
(d) (gal)(hour)/(miles)²

1.5.8 \( g(v) \) gives the fuel efficiency, in miles per gallon, of a car going a speed of \( v \) miles per hour. What is the practical meaning of \( g'(55) = -0.54 \)?

(a) When the car is going 55 mph, the rate of change of the fuel efficiency decreases to 0.54 miles/gal.
(b) When the car is going 55 mph, the rate of change of the fuel efficiency decreases by 0.54 miles/gal.
(c) If the car speeds up from 55 to 56 mph, then the fuel efficiency is 0.54 miles per gallon.
(d) If the car speeds up from 55 to 56 mph, then the car becomes less fuel efficient by 0.54 miles per gallon.
1.6  The second derivative

Preview Activity 1.7. The position of a car driving along a straight road at time \( t \) in minutes is given by the function \( y = s(t) \) that is pictured in Figure 1.8. The car’s position function has units measured in thousands of feet. For instance, the point \((2, 4)\) on the graph indicates that after 2 minutes, the car has traveled 4000 feet.

![Figure 1.8: The graph of \( y = s(t) \), the position of the car (measured in thousands of feet from its starting location) at time \( t \) in minutes.](image)

(a) In everyday language, describe the behavior of the car over the provided time interval. In particular, you should carefully discuss what is happening on each of the time intervals \([0, 1], [1, 2], [2, 3], [3, 4], \) and \([4, 5]\), plus provide commentary overall on what the car is doing on the interval \([0, 12]\).

(b) On the lefthand axes provided in Figure 1.9, sketch a careful, accurate graph of \( y = s'(t) \).

(c) What is the meaning of the function \( y = s'(t) \) in the context of the given problem? What can we say about the car’s behavior when \( s'(t) \) is positive? when \( s'(t) \) is zero? when \( s'(t) \) is negative?

(d) Rename the function you graphed in (b) to be called \( y = v(t) \). Describe the behavior of \( v \) in words, using phrases like “\( v \) is increasing on the interval . . .” and “\( v \) is constant on the interval . . .”

(e) Sketch a graph of the function \( y = v'(t) \) on the righthand axes provide in Figure 1.8. Write at least one sentence to explain how the behavior of \( v'(t) \) is connected to the graph of \( y = v(t) \).
Figure 1.9: Axes for plotting \( y = v(t) = s'(t) \) and \( y = v'(t) \).

Activity 1.15.

The position of a car driving along a straight road at time \( t \) in minutes is given by the function \( y = s(t) \) that is pictured in Figure 1.10. The car’s position function has units measured in thousands of feet. Remember that you worked with this function and sketched graphs of \( y = v(t) = s'(t) \) and \( y = v'(t) \) in Preview Activity 1.7.

(a) On what intervals is the position function \( y = s(t) \) increasing? decreasing? Why?

(b) On which intervals is the velocity function \( y = v(t) = s'(t) \) increasing? decreasing? neither? Why?

(c) Acceleration is defined to be the instantaneous rate of change of velocity, as the acceleration of an object measures the rate at which the velocity of the object is changing. Say that the car’s acceleration function is named \( a(t) \). How is \( a(t) \) computed from \( v(t) \)? How is \( a(t) \) computed from \( s(t) \)? Explain.

(d) What can you say about \( s'' \) whenever \( s' \) is increasing? Why?

(e) Using only the words increasing, decreasing, constant, concave up, concave down, and linear, complete the following sentences. For the position function \( s \) with velocity \( v \) and acceleration \( a \),

- on an interval where \( v \) is positive, \( s \) is ________________.
- on an interval where \( v \) is negative, \( s \) is ________________.
- on an interval where \( v \) is zero, \( s \) is ________________.
- on an interval where \( a \) is positive, \( v \) is ________________.
- on an interval where \( a \) is negative, \( v \) is ________________.
- on an interval where \( a \) is zero, \( v \) is ________________.
Figure 1.10: The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time $t$ in minutes.

- on an interval where $a$ is positive, $s$ is ________________.
- on an interval where $a$ is negative, $s$ is ________________.
- on an interval where $a$ is zero, $s$ is ________________.
Activity 1.16.

This activity builds on our experience and understanding of how to sketch the graph of $f'$ given the graph of $f$. Below, given the graph of a function $f$, sketch $f'$ on the first axes below, and then sketch $f''$ on the second set of axes. In addition, for each, write several careful sentences in the spirit of those in Activity 1.15 that connect the behaviors of $f$, $f'$, and $f''$. For instance, write something such as

$f'$ is __________ on the interval _____, which is connected to the fact that $f$ is __________ on the same interval _____, and $f''$ is __________
on the interval as well

but of course with the blanks filled in. Throughout, view the scale of the grid for the graph of $f$ as being $1 \times 1$, and assume the horizontal scale of the grid for the graph of $f'$ is identical to that for $f$. If you need to adjust the vertical scale on the axes for the graph of $f'$ or $f''$, you should label that accordingly.
Activity 1.17.

A potato is placed in an oven, and the potato’s temperature $F$ (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time $t$ is measured in minutes. In Activity 1.12, we computed approximations to $F'(30)$ and $F'(60)$ using central differences. Those values and more are provided in the second table below, along with several others computed in the same way.
(a) What are the units on the values of $F'(t)$?

(b) Use a central difference to estimate the value of $F''(30)$.

(c) What is the meaning of the value of $F''(30)$ that you have computed in (c) in terms of the potato’s temperature? Write several careful sentences that discuss, with appropriate units, the values of $F(30)$, $F'(30)$, and $F''(30)$, and explain the overall behavior of the potato’s temperature at this point in time.

(d) Overall, is the potato’s temperature increasing at an increasing rate, increasing at a constant rate, or increasing at a decreasing rate? Why?
1.6. THE SECOND DERIVATIVE

Voting Questions

1.6.1 The graph of \( y = f(x) \) is shown in figure 2.18. Which of the following is true for \( f \) on the interval shown?

i. \( f(x) \) is positive
ii. \( f(x) \) is increasing
iii. \( f'(x) \) is positive
iv. \( f'(x) \) is increasing
v. \( f''(x) \) is positive

(a) i, ii, and iii only
(b) ii, iii, and v only
(c) ii, iii, iv, and v only
(d) all are true
(e) the correct combination of true statements is not listed here

1.6.2 True or False: If \( f''(x) \) is positive, then \( f(x) \) is concave up.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.6.3 True or False: If \( f''(x) \) is positive, then \( f'(x) \) is increasing.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.6.4 True or False: If \( f'(x) \) is increasing, then \( f(x) \) is concave up.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
1.6.5 **True or False:** If the velocity of an object is constant, then its acceleration is zero.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.6.6 In Figure 2.21, the second derivative at points \(a\), \(b\), and \(c\), respectively, is

(a) +, 0, -
(b) -, 0, +
(c) -, 0, -
(d) +,0,+
(e) +,+,-
(f) -,+,+

1.6.7 In Figure 2.22, at \(x = 0\) the signs of the function and the first and second derivatives, in order, are

(a) +, 0, +
(b) +,0,-
(c) -, +, -
(d) +,+,+
(e) +,-,+ 
(f) +,+,+

1.6.8 Which of the following graphs could represent the second derivative of the function shown in Figure 2.25?
1.6.10 In *Star Trek: First Contact*, Worf almost gets knocked into space by the Borg. Assume he was knocked into space and his space suit was equipped with thrusters. Worf fires his thrusters for 1 second, which produces a constant acceleration in the positive direction. In the next second he turns off his thrusters. In the third second he fires his thruster producing a constant negative acceleration. The acceleration as a function of time is given in Figure 2.31. Which of the following graphs represent his position as a function of time?
1.6.11 The position of a moving car is given by the function $s(t) = 3t^2 + 3$, where $t$ is in seconds, and $s$ is in feet. What function gives the car’s acceleration?

(a) $a(t) = 3$
(b) $a(t) = 6t$
(c) $a(t) = 6$
(d) $a(t) = 6t + 3$
(e) $a(t) = 9$
1.7 Limits, Continuity, and Differentiability

**Preview Activity 1.8.** A function $f$ defined on $-4 < x < 4$ is given by the graph in Figure 1.12. Use the graph to answer each of the following questions. Note: to the right of $x = 2$, the graph of $f$ is exhibiting infinite oscillatory behavior.

(a) For each of the values $a = -3, -2, -1, 0, 1, 2, 3$, determine whether or not $\lim_{x \to a} f(x)$ exists. If the function has a limit $L$ at a given point, state the value of the limit using the notation $\lim_{x \to a} f(x) = L$. If the function does not have a limit at a given point, write a sentence to explain why.

(b) For each of the values of $a$ from part (a) where $f$ has a limit, determine the value of $f(a)$ at each such point. In addition, for each such $a$ value, does $f(a)$ have the same value as $\lim_{x \to a} f(x)$?

(c) For each of the values $a = -3, -2, -1, 0, 1, 2, 3$, determine whether or not $f'(a)$ exists. In particular, based on the given graph, ask yourself if it is reasonable to say that $f$ has a tangent line at $(a, f(a))$ for each of the given $a$-values. If so, visually estimate the slope of the tangent line to find the value of $f'(a)$.
Activity 1.18.

Consider a function that is piecewise-defined according to the formula

\[ f(x) = \begin{cases} 
3(x + 2) + 2 & \text{for } -3 < x < -2 \\
\frac{2}{3}(x + 2) + 1 & \text{for } -2 \leq x < -1 \\
\frac{2}{3}(x + 2) + 1 & \text{for } -1 < x < 1 \\
2 & \text{for } x = 1 \\
4 - x & \text{for } x > 1 
\end{cases} \]

Use the given formula to answer the following questions.

(a) For each of the values \(a = -2, -1, 0, 1, 2\), compute \(f(a)\).

(b) For each of the values \(a = -2, -1, 0, 1, 2\), determine \(\lim_{x \to a^-} f(x)\) and \(\lim_{x \to a^+} f(x)\).

(c) For each of the values \(a = -2, -1, 0, 1, 2\), determine \(\lim_{x \to a} f(x)\). If the limit fails to exist, explain why by discussing the left- and right-hand limits at the relevant \(a\)-value.

(d) For which values of \(a\) is the following statement true?

\[ \lim_{x \to a} f(x) \neq f(a) \]

(e) On the axes provided in Figure 1.13, sketch an accurate, labeled graph of \(y = f(x)\). Be sure to carefully use open circles (◦) and filled circles (●) to represent key points on the graph, as dictated by the piecewise formula.
1.7. LIMITS, CONTINUITY, AND DIFFERENTIABILITY

Activity 1.19.

This activity builds on your work in Preview Activity 1.8, using the same function \( f \) as given by the graph that is repeated in Figure 1.14.

![Graph of \( f(x) \)](image)

Figure 1.14: The graph of \( y = f(x) \) for Activity 1.19.

(a) At which values of \( a \) does \( \lim_{x \to a} f(x) \) not exist?

(b) At which values of \( a \) is \( f(a) \) not defined?

(c) At which values of \( a \) does \( f \) have a limit, but \( \lim_{x \to a} f(x) \neq f(a) \)?

(d) State all values of \( a \) for which \( f \) is not continuous at \( x = a \).

(e) Which condition is stronger, and hence implies the other: \( f \) has a limit at \( x = a \) or \( f \) is continuous at \( x = a \)? Explain, and hence complete the following sentence: “If \( f \) ___________ at \( x = a \), then \( f \) ___________ at \( x = a \),” where you complete the blanks with **has a limit** and **is continuous**, using each phrase once.

\(\triangleright\)
Activity 1.20.

In this activity, we explore two different functions and classify the points at which each is not differentiable. Let \( g \) be the function given by the rule \( g(x) = |x| \), and let \( f \) be the function that we have previously explored in Preview Activity 1.8, whose graph is given again in Figure 1.15.

(a) Reasoning visually, explain why \( g \) is differentiable at every point \( x \) such that \( x \neq 0 \).
(b) Use the limit definition of the derivative to show that \( g'(0) = \lim_{h \to 0} \frac{|h|}{h} \).
(c) Explain why \( g'(0) \) fails to exist by using small positive and negative values of \( h \).

![Figure 1.15: The graph of \( y = f(x) \) for Activity 1.20.](image)

(d) State all values of \( a \) for which \( f \) is not differentiable at \( x = a \). For each, provide a reason for your conclusion.
(e) True or false: if a function \( p \) is differentiable at \( x = b \), then \( \lim_{x \to b} p(x) \) must exist. Why?

\(<\)
1.7. LIMITS, CONTINUITY, AND DIFFERENTIABILITY

Voting Questions

1.7.1 A drippy faucet adds one milliliter to the volume of water in a tub at precisely one-second intervals. Let $f$ be the function that represents the volume of water in the tub at time $t$. Which of the following statements is correct?

(a) $f$ is a continuous function at every time $t$
(b) $f$ is continuous for all $t$ other than the precise instants when the water drips into the tub.
(c) $f$ is not continuous at any time $t$.
(d) There is not enough information to know where $f$ is continuous.

1.7.2 A drippy faucet adds one milliliter to the volume of water in a tub at precisely one second intervals. Let $g$ be the function that represents the volume of water in the tub as a function of the depth of the water, $x$, in the tub. Which of the following statements is correct?

(a) $g$ is a continuous function at every depth $x$.
(b) there are some values of $x$ at which $g$ is not continuous.
(c) $g$ is not continuous at any depth, $x$.
(d) not enough information is given to know where $g$ is continuous.

1.7.3 You know the following statement is true:

If $f(x)$ is a polynomial, then $f(x)$ is continuous.

Which of the following is also true?

(a) If $f(x)$ is not continuous, then it is not a polynomial.
(b) If $f(x)$ is continuous, then it is a polynomial.
(c) If $f(x)$ is not a polynomial, then it is not continuous.

1.7.4 True or False: You were once exactly 3 feet tall.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
1.7.5 **True or False:** At some time since you were born your weight in pounds equaled your height in inches.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.7.6 **True or False:** Along the Equator, there are two diametrically opposite sites that have exactly the same temperature at the same time.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.7.7 Suppose that during half-time at a basketball game the score of the home team was 36 points. **True or False:** There had to be at least one moment in the first half when the home team had exactly 25 points.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.7.8 At what point on the interval \([-7, 2]\) does the function \(f(x) = \frac{3x^2}{4e^x - 4}\) have a discontinuity?

(a) \(x = 0\)
(b) \(x = 1\)
(c) \(x = 3\)
(d) \(x = 4\)
(e) There is no discontinuity on this interval.

1.7.9 For what value of the constant \(c\) is the function \(f(x)\) continuous, if

\[
f(x) = \begin{cases} 
    cx + 9 & \text{if } x \in (-\infty, 5] \\
    cx^2 - 9 & \text{if } x \in (5, \infty)
\end{cases}
\]
1.7. LIMITS, CONTINUITY, AND DIFFERENTIABILITY

(a) \( c = \frac{-9}{5} \)

(b) \( c = \frac{9}{10} \)

(c) \( c = \frac{9}{25} \)

(d) This is not possible.

1.7.10 Your mother says “If you eat your dinner, you can have dessert.” You know this means, “If you don’t eat your dinner, you cannot have dessert.” Your calculus teacher says, “If \( f \) is differentiable at \( x \), \( f \) is continuous at \( x \).” You know this means

(a) if \( f \) is not continuous at \( x \), \( f \) is not differentiable at \( x \).

(b) if \( f \) is not differentiable at \( x \), \( f \) is not continuous at \( x \).

(c) knowing \( f \) is not continuous at \( x \), does not give us enough information to deduce anything about whether the derivative of \( f \) exists at \( x \).

1.7.11 If \( f'(a) \) exists, \( \lim_{x \to a} f(x) \)

(a) must exist, but there is not enough information to determine it exactly.

(b) equals \( f(a) \).

(c) equals \( f'(a) \).

(d) may not exist.

1.7.12 True or False: The function \( f(x) = x^{1/3} \) is continuous at \( x = 0 \).

(a) True, and I am very confident

(b) True, but I am not very confident

(c) False, but I am not very confident

(d) False, and I am very confident

1.7.13 True or False: If \( f(x) = x^{1/3} \) then there is a tangent line at (0,0).

(a) True, and I am very confident

(b) True, but I am not very confident

(c) False, but I am not very confident

(d) False, and I am very confident
True or False: If $f(x) = x^{1/3}$ then $f'(0)$ exists.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
1.8 The Tangent Line Approximation

Preview Activity 1.9. Consider the function \( y = g(x) = -x^2 + 3x + 2 \).

(a) Use the limit definition of the derivative to compute a formula for \( y = g'(x) \).

(b) Determine the slope of the tangent line to \( y = g(x) \) at the value \( x = 2 \).

(c) Compute \( g(2) \).

(d) Find an equation for the tangent line to \( y = g(x) \) at the point \((2, g(2))\). Write your result in point-slope form\(^1\).

(e) On the axes provided in Figure 1.16, sketch an accurate, labeled graph of \( y = g(x) \) along with its tangent line at the point \((2, g(2))\).

---

\(^1\)Recall that a line with slope \( m \) that passes through \((x_0, y_0)\) has equation \( y - y_0 = m(x - x_0) \), and this is the point-slope form of the equation.
Activity 1.21. 
Suppose it is known that for a given differentiable function \( y = g(x) \), its local linearization at the point where \( a = -1 \) is given by \( L(x) = -2 + 3(x + 1) \).

(a) Compute the values of \( L(-1) \) and \( L'(-1) \).

(b) What must be the values of \( g(-1) \) and \( g'(-1) \)? Why?

(c) Do you expect the value of \( g(-1.03) \) to be greater than or less than the value of \( g(-1) \)? Why?

(d) Use the local linearization to estimate the value of \( g(-1.03) \).

(e) Suppose that you also know that \( g''(-1) = 2 \). What does this tell you about the graph of \( y = g(x) \) at \( a = -1 \)?

(f) For \( x \) near \(-1\), sketch the graph of the local linearization \( y = L(x) \) as well as a possible graph of \( y = g(x) \) on the axes provided in Figure 1.17.

Figure 1.17: Axes for plotting \( y = L(x) \) and \( y = g(x) \).
Activity 1.22.

The circumference of a sphere was measured to be 71.0 cm with a possible error of 0.5 cm. In this activity we’ll use linear approximation to estimate the maximum error in the calculated surface area.

(a) Write the formula for the surface area of a sphere in terms of the radius, and write the formula for the circumference of a circle in terms of the radius.

(b) Fill in the blanks with the help of equation ??.

\[
\Delta C = \underline{\; \Delta r \;} \\
\Delta S = \underline{\; \Delta r \;} \\
\]

(c) Use your answer from part (b) along with the fact that \(\Delta C = 0.5\) and \(C = 71\) to calculate the error in surface area: \(\Delta S\).

(d) Estimate the relative error (fractional error) in the calculated surface area.

\(\triangleright\)
Activity 1.23.

Use linear approximation to approximate $\sqrt{4.1}$ using the following hints:

- Let $f(x) = \sqrt{x}$ and find the equation of the tangent line to $f(x)$ at a “nice” point near 4.1.
- Then use this to approximate $\sqrt{4.1}$.
Activity 1.24.

This activity concerns a function $f(x)$ about which the following information is known:

- $f$ is a differentiable function defined at every real number $x$
- $f(2) = -1$
- $y = f'(x)$ has its graph given in Figure 1.18

Your task is to determine as much information as possible about $f$ (especially near the value $a = 2$) by responding to the questions below.

(a) Find a formula for the tangent line approximation, $L(x)$, to $f$ at the point $(2, -1)$.

(b) Use the tangent line approximation to estimate the value of $f(2.07)$. Show your work carefully and clearly.

(c) Sketch a graph of $y = f''(x)$ on the righthand grid in Figure 1.18; label it appropriately.

(d) Is the slope of the tangent line to $y = f(x)$ increasing, decreasing, or neither when $x = 2$? Explain.

(e) Sketch a possible graph of $y = f(x)$ near $x = 2$ on the lefthand grid in Figure 1.18. Include a sketch of $y = L(x)$ (found in part (a)). Explain how you know the graph of $y = f(x)$ looks like you have drawn it.

(f) Does your estimate in (b) over- or under-estimate the true value of $f(2)$? Why?
1.8. Voting Questions

1.8.1 If \( e^{0.5} \) is approximated by using the tangent line to the graph of \( f(x) = e^x \) at \((0,1)\), and we know \( f'(0) = 1 \), the approximation is

(a) 0.5  
(b) 1 + \( e^{0.5} \)  
(c) 1 + 0.5

1.8.2 The line tangent to the graph of \( f(x) = \sin x \) at \((0,0)\) is \( y = x \). This implies that

(a) \( \sin(0.0005) \approx 0.0005 \)  
(b) The line \( y = x \) touches the graph of \( f(x) = \sin x \) at exactly one point, \((0,0)\).  
(c) \( y = x \) is the best straight line approximation to the graph of \( f \) for all \( x \).

1.8.3 The line \( y = 1 \) is tangent to the graph of \( f(x) = \cos x \) at \((0,1)\). This means that

(a) the only \( x \)-values for which \( y = 1 \) is a good estimate for \( y = \cos x \) are those that are close enough to 0.  
(b) tangent lines can intersect the graph of \( f \) infinitely many times.  
(c) the farther \( x \) is from 0, the worse the linear approximation is.  
(d) All of the above

1.8.4 Suppose that \( f''(x) < 0 \) for \( x \) near a point \( a \). Then the linearization of \( f \) at \( a \) is

(a) an over approximation  
(b) an under approximation  
(c) unknown without more information.

1.8.5 Peeling an orange changes its volume \( V \). What does \( \Delta V \) represent?

(a) the volume of the rind  
(b) the surface area of the orange  
(c) the volume of the “edible part” of the orange  
(d) \(-1 \times \) (the volume of the rind)
1.8.6 You wish to approximate $\sqrt{9.3}$. You know the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ where $x = 9$. What value do you put into the tangent line equation to approximate $\sqrt{9.3}$?

(a) $\sqrt{9.3}$
(b) 9
(c) 9.3
(d) 0.3

1.8.7 We can use a tangent line approximation to $\sqrt{x}$ to approximate square roots of numbers. If we do that for each of the square roots below, for which one would we get the smallest error?

(a) $\sqrt{4.2}$
(b) $\sqrt{4.5}$
(c) $\sqrt{9.2}$
(d) $\sqrt{9.5}$
(e) $\sqrt{16.2}$
(f) $\sqrt{16.5}$
1.8. THE TANGENT LINE APPROXIMATION
Chapter 2

Computing Derivatives

2.1 Elementary derivative rules

Preview Activity 2.1. Functions of the form \( f(x) = x^n \), where \( n = 1, 2, 3, \ldots \), are often called power functions.

(a) Use the limit definition of the derivative to find \( f'(x) \) for \( f(x) = x^2 \).

(b) Use the limit definition of the derivative to find \( f'(x) \) for \( f(x) = x^3 \).

(c) Use the limit definition of the derivative to find \( f'(x) \) for \( f(x) = x^4 \). (Hint: \( (a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \). Apply this rule to \( (x + h)^4 \) within the limit definition.)

(d) Based on your work in (a), (b), and (c), what do you conjecture is the derivative of \( f(x) = x^5 \)? Of \( f(x) = x^{13} \)?

(e) Conjecture a formula for the derivative of \( f(x) = x^n \) that holds for any positive integer \( n \). That is, given \( f(x) = x^n \) where \( n \) is a positive integer, what do you think is the formula for \( f'(x) \)?
Activity 2.1.

Use the three rules above to determine the derivative of each of the following functions. For each, state your answer using full and proper notation, labeling the derivative with its name. For example, if you are given a function $h(z)$, you should write “$h'(z) = $” or “$\frac{dh}{dz} =$” as part of your response.

(a) $f(t) = \pi$
(b) $g(z) = 7z$
(c) $h(w) = w^{3/4}$
(d) $p(x) = 3^{1/2}$
(e) $r(t) = (\sqrt{2})^t$
(f) $\frac{d}{dq}[q^{-1}]$
(g) $m(t) = \frac{1}{t^3}$
Activity 2.2.

Use only the rules for constant, power, and exponential functions, together with the Constant Multiple and Sum Rules, to compute the derivative of each function below with respect to the given independent variable. Note well that we do not yet know any rules for how to differentiate the product or quotient of functions. This means that you may have to do some algebra first on the functions below before you can actually use existing rules to compute the desired derivative formula. In each case, label the derivative you calculate with its name using proper notation such as $f'(x)$, $h'(z)$, $dr/dt$, etc.

(a) $f(x) = x^{5/3} - x^4 + 2^x$
(b) $g(x) = 14e^x + 3x^5 - x$
(c) $h(z) = \sqrt{z} + \frac{1}{z^4} + 5^z$
(d) $r(t) = \sqrt{53} t^7 - \pi e^t + e^4$
(e) $s(y) = (y^2 + 1)(y^2 - 1)$
(f) $q(x) = \frac{x^3 - x + 2}{x}$
(g) $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$
Activity 2.3.

Each of the following questions asks you to use derivatives to answer key questions about functions. Be sure to think carefully about each question and to use proper notation in your responses.

(a) Find the slope of the tangent line to \( h(z) = \sqrt{z} + \frac{1}{z} \) at the point where \( z = 4 \).

(b) A population of cells is growing in such a way that its total number in millions is given by the function \( P(t) = 2(1.37)^t + 32 \), where \( t \) is measured in days.
   
i. Determine the instantaneous rate at which the population is growing on day 4, and include units on your answer.
   
ii. Is the population growing at an increasing rate or growing at a decreasing rate on day 4? Explain.

(c) Find an equation for the tangent line to the curve \( p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12 \) at the point where \( a = -1 \).

(d) What is the difference between being asked to find the slope of the tangent line (asked in (a)) and the equation of the tangent line (asked in (c))?
Voting Questions

2.1.1 If \( f(x) = 2x^2 \), then what is \( f'(x) \)?

(a) \( 2x \)
(b) \( 2x^2 \)
(c) \( 4 \)
(d) \( 4x \)
(e) \( 4x^2 \)
(f) Cannot be determined from what we know

2.1.2 If \( f(x) = 7 \), then what is \( f'(x) \)?

(a) \( 7 \)
(b) \( 7x \)
(c) \( 0 \)
(d) \( 1 \)
(e) Cannot be determined from what we know

2.1.3 If \( f(x) = 2x^{2.5} \), then what is \( f'(x) \)?

(a) \( 2.5x^{2.5} \)
(b) \( 5x^{2.5} \)
(c) \( 2.5x^{1.5} \)
(d) \( 5x^{1.5} \)
(e) Cannot be determined from what we know

2.1.4 If \( f(x) = \pi^2 \), then what is \( f'(x) \)?

(a) \( 2\pi \)
(b) \( \pi^2 \)
(c) \( 0 \)
(d) \( 2 \)
(e) Cannot be determined from what we know
2.1.5 If \( f(x) = 3^x \), then what is \( f'(x) \)?

(a) \( x \cdot 3^{x-1} \)
(b) \( 3^x \)
(c) \( 3x^2 \)
(d) 0
(e) Cannot be determined from what we know

2.1.6 If \( f(x) = 4\sqrt{x} \), then what is \( f'(x) \)?

(a) \( 4\sqrt{x} \)
(b) \( 2\sqrt{x} \)
(c) \( 2x^{1/2} \)
(d) \( 4x^{-1/2} \)
(e) \( 2x^{-1/2} \)
(f) Cannot be determined from what we know

2.1.7 If \( f(t) = 3t^2 + 2t \), then what is \( f'(t) \)?

(a) \( 3t^2 + 2 \)
(b) \( 6t + 2 \)
(c) \( 9t^2 + 2t \)
(d) \( 9t + 2 \)
(e) Cannot be determined from what we know

2.1.8 Let \( f(x) = -16x^2 + 96x \). Find \( f'(2) \).

(a) 0
(b) 32
(c) 128
(d) \( f'(2) \) does not exist.

2.1.9 If \( a + b^2 = 3 \), find \( \frac{da}{db} \).

(a) \( \frac{da}{db} = 0 \)
2.1. ELEMENTARY DERIVATIVE RULES

(b) \( \frac{da}{db} = 2b \)
(c) \( \frac{da}{db} = -2b \)
(d) Cannot be determined from this expression

2.1.10 If \( r(q) = 4q^{-5} \), then what is \( r'(q) \)?

(a) \( 5q^{-5} \)
(b) \( -20q^{-4} \)
(c) \( -20q^{-5} \)
(d) \( -20q^{-6} \)
(e) Cannot be determined from what we know

2.1.11 If \( f(x) = x(x + 5) \), then what is \( f'(x) \)?

(a) \( x + 5 \)
(b) \( 1 \)
(c) \( 2x + 5 \)
(d) \( 2x \)
(e) Cannot be determined from what we know

2.1.12 If \( f(x) = \frac{2}{x^3} \), then what is \( f'(x) \)?

(a) \( \frac{2}{3x^3} \)
(b) \( \frac{-6}{x^4} \)
(c) \( 6x^{-2} \)
(d) \( -3x^{-4} \)
(e) Cannot be determined from what we know

2.1.13 If \( f(x) = x^2 + \frac{3}{x} \), then what is \( f'(x) \)?

(a) \( 2x - 3x^{-2} \)
(b) \( 2x + 3x^{-1} \)
(c) \( 2x - 3x^2 \)
(d) \( x^2 - 3x^{-1} \)
(e) Cannot be determined from what we know
2.1.14 If \( f(x) = 4\sqrt{x} + \frac{5}{x^2} \), then what is \( f'(x) \)?

(a) \( 2x^{-1/2} - 10x^{-3} \)
(b) \( 4x^{1/2} + 5x^{-2} \)
(c) \( 2x^{1/2} - 10x^{-3} \)
(d) \( 2x^{-1/2} + 10x^{-3} \)
(e) Cannot be determined from what we know

2.1.15 If \( f(x) = \frac{x^2 + 5x}{x} \), then what is \( f'(x) \)?

(a) \( 2x + 5 \)
(b) \( x + 5 \)
(c) \( 1 \)
(d) \( 0 \)
(e) Cannot be determined from what we know

2.1.16 If \( f(x) = \frac{x}{x^2 + 5x} \), then what is \( f'(x) \)?

(a) \( \frac{1}{2x+5} \)
(b) \( -x^{-2} \)
(c) \( \frac{1}{x} + \frac{1}{5} \)
(d) \( 1 \)
(e) Cannot be determined from what we know

2.1.17 If \( f(m) = am^2 + bm \), then what is \( f'(m) \)?

(a) \( m^2 + m \)
(b) \( 2am + b \)
(c) \( am \)
(d) \( 0 \)
(e) Cannot be determined from what we know

2.1.18 If \( p(q) = \frac{2q - 8}{q^2} \), then what is \( p'(2) \)?

(a) \( \frac{2}{2q} \)
2.1. ELEMENTARY DERIVATIVE RULES

(b) \(-2q^{-2} + 16q^{-3}\)
(c) \(\frac{1}{2}\)
(d) \(\frac{3}{2}\)
(e) 0
(f) Cannot be determined from what we know

2.1.19 If \(f(d) = ad^2 + bd + d + c\), then what is \(f'(d)\)?

(a) \(2ad + b + d\)
(b) \(2ad + b + 1\)
(c) \(2ad + b + c\)
(d) \(2ad + b\)
(e) \(2ad + b + 1 + c\)
(f) \(2ad + b + 2\)

2.1.20 If \(g(d) = ab^2 + 3c^3d + 5b^2c^2d^2\), then what is \(g''(d)\)?

(a) \(3c^3 + 10b^2c^2d\)
(b) \(10b^2c^2\)
(c) \(42 + 18cd\)
(d) \(2ab + 9c^2d + 40bcd\)
(e) Cannot be determined from what we know

2.1.21 Find the equation of the line that is tangent to the function \(f(x) = 3x^2\) when \(x = 2\). Recall that this line not only has the same slope as \(f(x)\) at \(x = 2\), but also has the same value of \(y\) when \(x = 2\).

(a) \(y = 12x - 12\)
(b) \(y = 6x\)
(c) \(y = 3x + 6\)
(d) \(y = 12x\)
(e) \(y = 6x + 6\)

2.1.22 Which is the equation of the line tangent to \(y = x^2\) at \(x = 4\)?
2.1.23 A ball is thrown into the air and its height \( h \) (in meters) after \( t \) seconds is given by the function \( h(t) = 10 + 20t - 5t^2 \). When the ball reaches its maximum height, its velocity will be zero. At what time will the ball reach its maximum height?

(a) \( t = 0 \) seconds  
(b) \( t = 1 \) second  
(c) \( t = 2 \) seconds  
(d) \( t = 3 \) seconds  
(e) \( t = 4 \) seconds

2.1.24 A ball is thrown into the air and its height \( h \) (in meters) after \( t \) seconds is given by the function \( h(t) = 10 + 20t - 5t^2 \). When the ball reaches its maximum height, its velocity will be zero. What will be the ball’s maximum height?

(a) \( h = 10 \) meters  
(b) \( h = 20 \) meters  
(c) \( h = 30 \) meters  
(d) \( h = 40 \) meters  
(e) \( h = 50 \) meters

2.1.25 Suppose a stone is thrown vertically upward with an initial velocity of 64 ft/s from a bridge 96 ft above a river. By Newton’s laws of motion, the position of the stone (measured as the height above the ground) after \( t \) seconds is \( s(t) = -16t^2 + 64t + 96 \). How many seconds after it is thrown will the stone reach its maximum height?

(a) \( 2 - \sqrt{10} \) s  
(b) 2 s  
(c) \( 2 + \sqrt{10} \) s  
(d) 4 s

2.1.26 \( \frac{d}{dx} (e^x) \) is
2.1. ELEMENTARY DERIVATIVE RULES

(a) $xe^{x-1}$
(b) $e^x$
(c) $e^x \ln x$
(d) 0
(e) Cannot be determined from what we know

2.1.27 $\frac{d}{dx} (5^x)$ is

(a) $x5^{x-1}$
(b) $5^x$
(c) $5^x \ln x$
(d) $5^x \ln 5$
(e) Cannot be determined from what we know

2.1.28 $\frac{d}{dx} (x^e)$ is

(a) $ex^{e-1}$
(b) $x^e$
(c) $x^e \ln x$
(d) $ex$
(e) Cannot be determined from what we know

2.1.29 $\frac{d}{dx} (e^7)$ is

(a) $7e^6$
(b) $e^7$
(c) $e^7 \ln 7$
(d) 0
(e) Cannot be determined from what we know

2.1.30 $\frac{d}{dx} (3e^x)$ is

(a) $3xe^{x-1}$
(b) $3e^x$
(c) $e^x \ln 3$
(d) 3
(e) Cannot be determined from what we know

2.1.31 \( \frac{d}{dx} (2 \cdot 5^x) \) is

(a) \( 10^x \)
(b) \( 2 \cdot 5^x \)
(c) \( 10^x \ln 10 \)
(d) \( 2 \cdot 5^x \ln 5 \)
(e) \( 10^x \ln 5 \)
(f) Cannot be determined from what we know

2.1.32 \( \frac{d}{dx} (xe^x) \) is

(a) \( x^2e^{x^2-1} \)
(b) \( xe^x \)
(c) \( e^x \ln x \)
(d) Cannot be determined from what we know

2.1.33 If \( \ln x - y = 0 \), find \( \frac{dx}{dy} \).

(a) \( \frac{dx}{dy} = e^x \)
(b) \( \frac{dx}{dy} = e^{-x} \)
(c) \( \frac{dx}{dy} = e^y \)
(d) \( \frac{dx}{dy} = e^{-y} \)
(e) Cannot be determined from this expression

2.1.34 \( \frac{d}{dx} (e^{x+2}) \) is

(a) \( (x + 2)e^{x+1} \)
(b) \( e^2e^x \)
(c) \( e^2 \)
(d) 0
(e) Cannot be determined from what we know
2.1. ELEMENTARY DERIVATIVE RULES

2.1.35 $\frac{d}{dx} (e^{2x})$ is

(a) $e^{2x}$
(b) $e^2 e^x$
(c) 0
(d) Cannot be determined from what we know

2.1.36 If $u = 5^v$, find $\frac{d^2 u}{dv^2}$.

(a) $\frac{d^2 u}{dv^2} = 0$
(b) $\frac{d^2 u}{dv^2} = 5^v$
(c) $\frac{d^2 u}{dv^2} = 5^v \ln 5$
(d) $\frac{d^2 u}{dv^2} = 5^v (\ln 5)^2$
(e) $\frac{d^2 u}{dv^2} = v(v - 1)5^{v-2}$
(f) Cannot be determined from what we know

2.1.37 If $u = ve^w + xy^v$, find $\frac{du}{dv}$.

(a) $\frac{du}{dv} = e^w + xy^v \ln y$
(b) $\frac{du}{dv} = ve^w + xy^v \ln y$
(c) $\frac{du}{dv} = e^w + xy^v \ln v$
(d) $\frac{du}{dv} = ve^w + xy^v \ln v$
(e) Cannot be determined from what we know

2.1.38 Find the equation of the line that is tangent to the function $g(x) = 2e^x$ at $x = 1$.

(a) $y = 2e^x x$
(b) $y = 2ex$
(c) $y = 2e^x x + 2e$
(d) $y = 2ex + 2e$
(e) None of the above
2.2 The sine and cosine functions

Preview Activity 2.2. Consider the function \( g(x) = 2^x \), which is graphed in Figure 2.1.

(a) At each of \( x = -2, -1, 0, 1, 2 \), use a straightedge to sketch an accurate tangent line to \( y = g(x) \).

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point in (a).

(c) Use the limit definition of the derivative to estimate \( g'(0) \) by using small values of \( h \), and compare the result to your visual estimate for the slope of the tangent line to \( y = g(x) \) at \( x = 0 \) in (b).

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of \( y = g'(x) \) on the axes adjacent to the graph of \( y = g(x) \).

(e) Write at least one sentence that explains why it is reasonable to think that \( g'(x) = cg(x) \), where \( c \) is a constant. In addition, calculate \( \ln(2) \), and then discuss how this value, combined with your work above, reasonably suggests that \( g'(x) = 2^x \ln(2) \).

Figure 2.1: At left, the graph of \( y = g(x) = 2^x \). At right, axes for plotting \( y = g'(x) \).
Activity 2.4.

Consider the function \( f(x) = \sin(x) \), which is graphed in Figure 2.2 below. Note carefully that the grid in the diagram does not have boxes that are 1 \times 1, but rather approximately 1.57 \times 1, as the horizontal scale of the grid is \( \pi/2 \) units per box.

(a) At each of \( x = -2\pi, -3\pi/2, -\pi, -\pi/2, 0, \pi/2, \pi, 3\pi/2, 2\pi \), use a straightedge to sketch an accurate tangent line to \( y = f(x) \).

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Pay careful attention to the scale of the grid.

(c) Use the limit definition of the derivative to estimate \( f'(0) \) by using small values of \( h \), and compare the result to your visual estimate for the slope of the tangent line to \( y = f(x) \) at \( x = 0 \) in (b). Using periodicity, what does this result suggest about \( f'(2\pi) \)? about \( f'(-2\pi) \)?

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of \( y = f'(x) \) on the axes adjacent to the graph of \( y = f(x) \).

(e) What familiar function do you think is the derivative of \( f(x) = \sin(x) \)?

![Graph of \( y = f(x) = \sin(x) \).](image)

Figure 2.2: At left, the graph of \( y = f(x) = \sin(x) \).
Activity 2.5.

Consider the function \( g(x) = \cos(x) \), which is graphed in Figure 2.3 below. Note carefully that the grid in the diagram does not have boxes that are 1 \( \times \) 1, but rather approximately \( 1.57 \times 1 \), as the horizontal scale of the grid is \( \pi/2 \) units per box.

Figure 2.3: At left, the graph of \( y = g(x) = \cos(x) \).

(a) At each of \( x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \), use a straightedge to sketch an accurate tangent line to \( y = g(x) \).

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Again, note the scale of the axes and grid.

(c) Use the limit definition of the derivative to estimate \( g'(\frac{\pi}{2}) \) by using small values of \( h \), and compare the result to your visual estimate for the slope of the tangent line to \( y = g(x) \) at \( x = \frac{\pi}{2} \) in (b). Using periodicity, what does this result suggest about \( g'(-\frac{3\pi}{2}) \)? Can symmetry on the graph help you estimate other slopes easily?

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of \( y = g'(x) \) on the axes adjacent to the graph of \( y = g(x) \).

(e) What familiar function do you think is the derivative of \( g(x) = \cos(x) \)?

\(<\)
Activity 2.6.

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

(a) Determine the derivative of \( h(t) = 3 \cos(t) - 4 \sin(t) \).

(b) Find the exact slope of the tangent line to \( y = f(x) = 2x + \frac{\sin(x)}{2} \) at the point where \( x = \frac{\pi}{6} \).

(c) Find the equation of the tangent line to \( y = g(x) = x^2 + 2 \cos(x) \) at the point where \( x = \frac{\pi}{2} \).

(d) Determine the derivative of \( p(z) = z^4 + 4z^2 + 4 \cos(z) - \sin\left(\frac{\pi}{2}\right) \).

(e) The function \( P(t) = 24 + 8 \sin(t) \) represents a population of a particular kind of animal that lives on a small island, where \( P \) is measured in hundreds and \( t \) is measured in decades since January 1, 2010. What is the instantaneous rate of change of \( P \) on January 1, 2030? What are the units of this quantity? Write a sentence in everyday language that explains how the population is behaving at this point in time.
2.2.1 \( \frac{d}{dx} (-3 \sin x) \) is
(a) \( \cos x \)
(b) \( -3 \sin x \)
(c) \( 3 \cos x \)
(d) \( -3 \cos x \)

2.2.2 \( \frac{d}{dx} \frac{\cos x}{2x} \) is
(a) \( (\sin x)/25 \)
(b) \( -\sin x \)
(c) \( (-\sin x)/25 \)
(d) \( (-\cos x)/25 \)

2.2.3 The 4th derivative of \( \sin x \) is
(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
(d) \( -\cos x \)

2.2.4 The 10th derivative of \( \sin x \) is
(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
(d) \( -\cos x \)

2.2.5 The 100th derivative of \( \sin x \) is
(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
2.2. THE SINE AND COSINE FUNCTIONS

(d) $-\cos x$

2.2.6 The 41st derivative of $\sin x$ is

(a) $\sin x$
(b) $\cos x$
(c) $-\sin x$
(d) $-\cos x$

2.2.7 The equation of the line tangent to the graph of $\cos x$ at $x = 0$ is

(a) $y = 1$
(b) $y = 0$
(c) $y = \cos x$
(d) $y = x$

2.2.8 The 30th derivative of $\cos x$ is

(a) $\sin x$
(b) $\cos x$
(c) $-\sin x$
(d) $-\cos x$
2.3 The product and quotient rules

Preview Activity 2.3. Let \( u \) and \( v \) be the functions defined by \( u(t) = 2t^2 \) and \( v(t) = t^3 + 4t \).

(a) Determine \( u'(t) \) and \( v'(t) \).

(b) Let \( p(t) = 2t^2(t^3 + 4t) \) and observe that \( p(t) = u(t) \cdot v(t) \). Rewrite the formula for \( p \) by distributing the \( 2t^2 \) term. Then, compute \( p'(t) \) using the sum and constant multiple rules.

(c) True or false: \( p'(t) = u'(t) \cdot v'(t) \).

(d) Let \( q(t) = \frac{t^3 + 4t}{2t^2} \) and observe that \( q(t) = \frac{v(t)}{u(t)} \). Rewrite the formula for \( q \) by dividing each term in the numerator by the denominator and simplify to write \( q \) as a sum of constant multiples of powers of \( t \). Then, compute \( q'(t) \) using the sum and constant multiple rules.

(e) True or false: \( q'(t) = \frac{v'(t)}{u'(t)} \).
Activity 2.7.

Use the product rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. That is, if you’re given a formula for $f(x)$, clearly label the formula you find for $f'(x)$. It is not necessary to algebraically simplify any of the derivatives you compute.

(a) Let $m(w) = 3w^{17} + 4w$. Find $m'(w)$.

(b) Let $h(t) = (\sin(t) + \cos(t))t^4$. Find $h'(t)$.

(c) Determine the slope of the tangent line to the curve $y = f(x)$ at the point where $a = 1$ if $f$ is given by the rule $f(x) = e^x \sin(x)$.

(d) Find the tangent line approximation $L(x)$ to the function $y = g(x)$ at the point where $a = -1$ if $g$ is given by the rule $g(x) = (x^2 + x)^2$. 

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Activity 2.8.

Use the quotient rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. That is, if you’re given a formula for $f(x)$, clearly label the formula you find for $f'(x)$. It is not necessary to algebraically simplify any of the derivatives you compute.

(a) Let $r(z) = \frac{3z}{z^4 + 1}$. Find $r'(z)$.

(b) Let $v(t) = \frac{\sin(t)}{\cos(t) + t^2}$. Find $v'(t)$.

(c) Determine the slope of the tangent line to the curve $R(x) = \frac{x^2 - 2x - 8}{x^2 - 9}$ at the point where $x = 0$.

(d) When a camera flashes, the intensity $I$ of light seen by the eye is given by the function

$$I(t) = \frac{100t}{e^t},$$

where $I$ is measured in candles and $t$ is measured in milliseconds. Compute $I'(0.5)$, $I'(2)$, and $I'(5)$; include appropriate units on each value; and discuss the meaning of each.
Activity 2.9.

Use relevant derivative rules to answer each of the questions below. Throughout, be sure to use proper notation and carefully label any derivative you find by name.

(a) Let \( f(r) = (5r^3 + \sin(r))(4r - 2\cos(r)) \). Find \( f'(r) \).

(b) Let \( p(t) = \frac{\cos(t)}{t^6 \cdot 6^t} \). Find \( p'(t) \).

(c) Let \( g(z) = 3z^7 e^z - 2z^2 \sin(z) + \frac{z}{z^2 + 1} \). Find \( g'(z) \).

(d) A moving particle has its position in feet at time \( t \) in seconds given by the function

\[
    s(t) = \frac{3\cos(t) - \sin(t)}{e^t}.
\]

Find the particle’s instantaneous velocity at the moment \( t = 1 \).

(e) Suppose that \( f(x) \) and \( g(x) \) are differentiable functions and it is known that \( f(3) = -2 \), \( f'(3) = 7 \), \( g(3) = 4 \), and \( g'(3) = -1 \). If \( p(x) = f(x) \cdot g(x) \) and \( q(x) = \frac{f(x)}{g(x)} \), calculate \( p'(3) \) and \( q'(3) \).
2.3. THE PRODUCT AND QUOTIENT RULES

Voting Questions

2.3.1 \( \frac{d}{dx} (x^2 e^x) = \)

(a) \( 2xe^x \)
(b) \( x^2e^x \)
(c) \( 2xe^x + x^2 e^{x-1} \)
(d) \( 2xe^x + x^2 e^x \)

2.3.2 \( \frac{d}{dx} (xe^x) = \)

(a) \( xe^x + x^2 e^x \)
(b) \( e^x + xe^x \)
(c) \( 2xe^x + xe^x \)
(d) \( e^x \)

2.3.3 \( \frac{d}{dt} ((t^2 + 3) e^t) = \)

(a) \( 2te^t + (t^2 + 3) e^t \)
(b) \( (2t + 3) e^t + (t^2 + 3) e^t \)
(c) \( 2te^t \)
(d) \( 2te^t + t^2 e^t \)
(e) \( (t^2 + 3) e^t \)

2.3.4 \( \frac{d}{dx} (x^3 4^x) = \)

(a) \( 3x^2 4^x \ln 4 \)
(b) \( x^3 4^x + x^3 4^x \ln 4 \)
(c) \( 3x^2 4^x + x^3 4^x \)
(d) \( 3x^2 4^x + x^3 4^x \ln 4 \)

2.3.5 When differentiating a constant multiple of a function (like \( 3x^2 \)) the Constant Multiple Rule tells us \( \frac{d}{dx} cf(x) = c \frac{d}{dx} f(x) \) and the Product Rule says \( \frac{d}{dx} cf(x) = c \frac{d}{dx} f(x) + f(x) \frac{d}{dx} c \). Do these two rules agree?

(a) Yes, they agree.
2.3. THE PRODUCT AND QUOTIENT RULES

(b) No, they do not agree.

2.3.6 \( \frac{d}{dx} xe^x = \)

(a) \( e^x + xe^x \)
(b) \( x^2e^x \)
(c) \( \frac{xe^x - e^x}{x^4} \)
(d) \( \frac{x^2e^x - e^x}{e^{2x}} \)

2.3.7 \( \frac{d}{dx} \frac{x^{1.5}}{3^x} = \)

(a) \( 1.5x^{0.5} - 3^x \ln 3 \)
(b) \( 1.5x^{0.5} \cdot (x^{-1} - 3^x \ln 3) \)
(c) \( 1.5x^{0.5}x^{-1} - 3^x \ln 3 \)
(d) \( 1.5x^{0.5} + x^{-1.5}x \ln 3 \)

2.3.8 If \( e^a - \frac{b}{a^2} = 5 \), find \( \frac{db}{da} \).

(a) \( \frac{db}{da} = e^a \)
(b) \( \frac{db}{da} = a^2e^a \)
(c) \( \frac{db}{da} = a^2e^a - 5a^2 \)
(d) \( \frac{db}{da} = 2ae^a + a^2e^a - 10a \)
(e) \( \frac{db}{da} = 2ae^a + a^2e^a - 10ae^a - 5a^2e^a \)
(f) Cannot be determined from this expression

2.3.9 \( \frac{d}{dx} (25x^2e^x) = \)

(a) \( 50x^2e^x + 25x^2e^x \)
(b) \( 25xe^x + 25x^2e^x \)
(c) \( 50xe^x + 25x^2e^x \)
(d) \( 50xe^x + 25xe^x \)

2.3.10 \( \frac{d}{dx} \frac{3t+1}{3t+2} = \)
2.3. THE PRODUCT AND QUOTIENT RULES

(a) \[ \frac{3(5t+2)-(3t+1)5}{(5t+2)^2} \]
(b) \[ \frac{3(5t+2)-(3t+1)5}{(5t+1)^2} \]
(c) \[ \frac{(3t+1)(5t+2)-(3t+1)5}{(5t+2)^2} \]
(d) \[ \frac{3(5t+2)-(3t+1)(5t+2)}{(5t+2)^2} \]

2.3.11 \( \frac{d}{dt} \sqrt{t} = \)

(a) \[ \frac{1}{2} t^{-1/2} - 2t \]
(b) \[ \frac{1}{2} t^{-1/2} - 2t \sqrt{t} \]
(c) \[ \frac{1}{2} t^{-1/2} (t^2+1)^2 - 2t \sqrt{t} \]
(d) \[ \frac{t^{-1/2} (t^2+1) - 2t \sqrt{t}}{(t^2+1)^2} \]

2.3.12 If \( f(3) = 2, f'(3) = 4, g(3) = 1, g'(3) = 3 \), and \( h(x) = f(x)g(x) \), then what is \( h'(3) \)?

(a) 2
(b) 10
(c) 11
(d) 12
(e) 14

2.3.13 If \( f(3) = 2, f'(3) = 4, g(3) = 1, g'(3) = 3 \), and \( h(x) = \frac{f(x)}{g(x)} \), then what is \( h'(3) \)?

(a) \(-2\)
(b) 2
(c) \(-\frac{2}{9}\)
(d) \(\frac{2}{9}\)
(e) 5

2.3.14 If \( h = \frac{ab^2e^b}{c^3} \) then what is \( \frac{dh}{db} ? \)

(a) \(\frac{2abe^b}{c^3}\)
(b) \(\frac{2abe^b}{3c^2}\)
2.3.15 My old uncle Stanley has a collection of rare and valuable books: He has a total of 4,000 books, that are worth an average of $60 each. His books are rising in value over time, so that each year, the average price per book goes up by $0.50. However he also has to sell 30 books per year in order to pay for his snowboarding activities. The value of the collection is

(a) increasing by approximately $240,000 per year.
(b) increasing by approximately $2000 per year.
(c) increasing by approximately $200 per year.
(d) decreasing by approximately $1,800 per year.
(e) decreasing by approximately $119,970 per year.

2.3.16 The functions \( f(x) \) and \( h(x) \) are plotted below. The function \( g = 2fh \). What is \( g'(2) \)?

(a) \( g'(2) = -1 \)
(b) \( g'(2) = 2 \)
(c) \( g'(2) = 4 \)
(d) \( g'(2) = 32 \)

2.3.17 The 4th derivative of \( \cos x \) is

(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
(d) \( -\cos x \)
2.3.18 If \( f(x) = \frac{x}{\sin x} \), then \( f'(x) = \)

(a) \( \frac{\sin x - x \cos x}{\sin^2 x} \)
(b) \( \frac{\sin x - x \cos x}{\cos^2 x} \)
(c) \( \frac{x \cos x - x \sin x}{\sin^2 x} \)
(d) \( \frac{\cos x - x \cos x}{\sin^2 x} \)

2.3.19 If \( f(x) = \sin x \cos x \), then \( f'(x) = \)

(a) \( 1 - 2 \sin^2 x \)
(b) \( 2 \cos^2 x - 1 \)
(c) \( \cos 2x \)
(d) All of the above
(e) None of the above

2.3.20 \( \frac{d}{dx}(e^x \sin x) \) is

(a) \( e^x \cos x \)
(b) \( e^x \sin x \)
(c) \( e^x \cos x + e^x \sin x \)
(d) \( e^x \sin x - e^x \cos x \)
2.4 Derivatives of other trigonometric functions

Preview Activity 2.4. Consider the function $f(x) = \tan(x)$, and remember that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

(a) What is the domain of $f$?

(b) Use the quotient rule to show that one expression for $f'(x)$ is

$$f'(x) = \frac{\cos(x) \cos(x) + \sin(x) \sin(x)}{\cos^2(x)}.$$

(c) What is the Fundamental Trigonometric Identity? How can this identity be used to find a simpler form for $f'(x)$?

(d) Recall that $\sec(x) = \frac{1}{\cos(x)}$. How can we express $f'(x)$ in terms of the secant function?

(e) For what values of $x$ is $f'(x)$ defined? How does this set compare to the domain of $f$?

\[\square\]
Activity 2.10.
Let \( h(x) = \sec(x) \) and recall that \( \sec(x) = \frac{1}{\cos(x)} \).

(a) What is the domain of \( h \)?

(b) Use the quotient rule to develop a formula for \( h'(x) \) that is expressed completely in terms of \( \sin(x) \) and \( \cos(x) \).

(c) How can you use other relationships among trigonometric functions to write \( h'(x) \) only in terms of \( \tan(x) \) and \( \sec(x) \)?

(d) What is the domain of \( h' \)? How does this compare to the domain of \( h \)?
Activity 2.11.

Let \( p(x) = \csc(x) \) and recall that \( \csc(x) = \frac{1}{\sin(x)} \).

(a) What is the domain of \( p \)?

(b) Use the quotient rule to develop a formula for \( p'(x) \) that is expressed completely in terms of \( \sin(x) \) and \( \cos(x) \).

(c) How can you use other relationships among trigonometric functions to write \( p'(x) \) only in terms of \( \cot(x) \) and \( \csc(x) \)?

(d) What is the domain of \( p' \)? How does this compare to the domain of \( p \)?
Activity 2.12.

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

(a) Let \( f(x) = 5 \sec(x) - 2 \csc(x) \). Find the slope of the tangent line to \( f \) at the point where \( x = \frac{\pi}{3} \).

(b) Let \( p(z) = z^2 \sec(z) - z \cot(z) \). Find the instantaneous rate of change of \( p \) at the point where \( z = \frac{\pi}{4} \).

(c) Let \( h(t) = \frac{\tan(t)}{t^2 + 1} - 2e^t \cos(t) \). Find \( h'(t) \).

(d) Let \( g(r) = \frac{r \sec(r)}{5r} \). Find \( g'(r) \).

(e) When a mass hangs from a spring and is set in motion, the object’s position oscillates in a way that the size of the oscillations decrease. This is usually called a damped oscillation. Suppose that for a particular object, its displacement from equilibrium (where the object sits at rest) is modeled by the function

\[
s(t) = \frac{15 \sin(t)}{e^t}.
\]

Assume that \( s \) is measured in inches and \( t \) in seconds. Sketch a graph of this function for \( t \geq 0 \) to see how it represents the situation described. Then compute \( ds/dt \), state the units on this function, and explain what it tells you about the object’s motion. Finally, compute and interpret \( s'(2) \).
Voting Questions

2.4.1 If \( f(x) = \tan x \), then \( f'(x) = \)

(a) \( \sec^2 x \)
(b) \( \cot x \)
(c) \( - \cot x \)
(d) All of the above
(e) None of the above
2.5 The chain rule

**Preview Activity 2.5.** For each function given below, identify its fundamental algebraic structure. In particular, is the given function a sum, product, quotient, or composition of basic functions? If the function is a composition of basic functions, state a formula for the inner function \( u \) and the outer function \( f \) so that the overall composite function can be written in the form \( f(u(x)) \). If the function is a sum, product, or quotient of basic functions, use the appropriate rule to determine its derivative.

(a) \( h(x) = \tan(2^x) \)

(b) \( p(x) = 2^x \tan(x) \)

(c) \( r(x) = (\tan(x))^2 \)

(d) \( m(x) = e^{\tan(x)} \)

(e) \( w(x) = \sqrt{x} + \tan(x) \)

(f) \( z(x) = \sqrt{\tan(x)} \)
Activity 2.13.
For each function given below, identify an inner function $u$ and outer function $f$ to write the function in the form $f(u(x))$. Then, determine $f'(x)$, $u'(x)$, and $f''(u(x))$, and finally apply the chain rule to determine the derivative of the given function.

(a) $h(x) = \cos(x^4)$
(b) $p(x) = \sqrt{\tan(x)}$
(c) $s(x) = 2\sin(x)$
(d) $z(x) = \cot^5(x)$
(e) $m(x) = (\sec(x) + e^x)^9$
Activity 2.14.

For each of the following functions, find the function’s derivative. State the rule(s) you use, label relevant derivatives appropriately, and be sure to clearly identify your overall answer.

(a) \( p(r) = 4\sqrt{r^6} + 2e^r \)

(b) \( m(v) = \sin(v^2) \cos(v^3) \)

(c) \( h(y) = \frac{\cos(10y)}{e^{4y} + 1} \)

(d) \( s(z) = 2z^2 \sec(z) \)

(e) \( c(x) = \sin(e^{x^2}) \)
Activity 2.15.

Use known derivative rules, including the chain rule, as needed to answer each of the following questions.

(a) Find an equation for the tangent line to the curve \( y = \sqrt{e^x + 3} \) at the point where \( x = 0 \).

(b) If \( s(t) = \frac{1}{(t^2 + 1)^3} \) represents the position function of a particle moving horizontally along an axis at time \( t \) (where \( s \) is measured in inches and \( t \) in seconds), find the particle’s instantaneous velocity at \( t = 1 \). Is the particle moving to the left or right at that instant?

(c) At sea level, air pressure is 30 inches of mercury. At an altitude of \( h \) feet above sea level, the air pressure, \( P \), in inches of mercury, is given by the function

\[
P = 30e^{-0.000323h}.
\]

Compute \( dP/dh \) and explain what this derivative function tells you about air pressure, including a discussion of the units on \( dP/dh \). In addition, determine how fast the air pressure is changing for a pilot of a small plane passing through an altitude of 1000 feet.

(d) Suppose that \( f(x) \) and \( u(x) \) are differentiable functions and that the following information about them is known:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( u(x) )</th>
<th>( u'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>(2)</td>
<td>(-5)</td>
<td>(-3)</td>
<td>(4)</td>
</tr>
<tr>
<td>(2)</td>
<td>(-3)</td>
<td>(4)</td>
<td>(-1)</td>
<td>(2)</td>
</tr>
</tbody>
</table>

If \( C(x) \) is a function given by the formula \( f(u(x)) \), determine \( C'(2) \). In addition, if \( D(x) \) is the function \( f(f(x)) \), find \( D'(-1) \).
Voting Questions

2.5.1 \( \frac{d}{dx} \sin(\cos x) \) is

(a) \(- \cos x \cos(\cos x)\)
(b) \(- \sin x \cos(\sin x)\)
(c) \(- \sin x \sin(\cos x)\)
(d) \(- \sin x \cos(\cos x)\)

2.5.2 We know that \( \frac{d}{dx} \sin x = \cos x \). **True or False**: \( \frac{d}{dx} \sin(2x) = \cos(2x) \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.5.3 \( \frac{d}{dx} (10 \sin (12x)) \) is

(a) 120 \cos(12x)
(b) 10 \cos(12x)
(c) 120 \sin(12x)
(d) \(-120 \cos(12x)\)

2.5.4 \( \frac{d}{dx} (\sin(x^2 + 5)) \) is

(a) \cos(x^2 + 5)
(b) \sin(2x + 5)
(c) 2x \sin(x^2 + 5)
(d) 2x \cos(x^2 + 5)

2.5.5 \( \frac{d}{dx} (\sin^2(ax)) \) is

(a) 2 \sin(ax)
(b) 2 \cos(ax)
(c) 2a \sin(ax)
(d) 2a \sin(ax) \cos(ax)
2.5. THE CHAIN RULE

2.5.6 \( \frac{d}{dx} (\sin x + e^{\sin x}) \) is

(a) \( \cos x + e^{\cos x} \)
(b) \( \cos x + e^{\sin x} \)
(c) \( \cos x + e^{\sin x} \cos x \)
(d) None of the above

2.5.7 The equation of the line tangent to the graph of \( 2 \sin 3x \) at \( x = \frac{\pi}{3} \) is

(a) \( y = 6x - 2\pi \)
(b) \( y = 6x \cos 3x - 2\pi \)
(c) \( y = -6x + 2\pi \)
(d) \( y = -6x + 2\pi - 1 \)

2.5.8 \( \frac{d}{dx} (x^2 + 5)^{100} \) is

(a) \( 100(x^2 + 5)^{99} \)
(b) \( 100x(x^2 + 5)^{99} \)
(c) \( 200x(x^2 + 5)^{99} \)
(d) \( 200x(2x + 5)^{99} \)

2.5.9 \( \frac{d}{dx} e^{3x} \) is

(a) \( 3e^{3x} \)
(b) \( e^{3x} \)
(c) \( 3xe^{3x} \)
(d) \( 3e^{3} \)

2.5.10 \( \frac{d}{dx} \sqrt{1-x} \) is

(a) \( \frac{1}{2}(1-x)^{-1/2} \)
(b) \( -\frac{1}{2}(1-x)^{-1/2} \)
(c) \( -(1-x)^{-1/2} \)
(d) \( -\frac{1}{2}(1-x)^{1/2} \)
2.5.11 \( \frac{d}{dx}e^{x^2} = \)
(a) \( x^2e^{2x} \)
(b) \( xe^{x^2} \)
(c) \( 2xe^{x^2} \)
(d) \( xe^{x^2} \)

2.5.12 \( \frac{d}{dx}3^{4x+1} = \)
(a) \( 4 \cdot 3^{4x+1} \ln 4 \)
(b) \( 4 \cdot 3^{4x+1} \ln 3 \)
(c) \( (4x + 1) \cdot 3^{4x+1} \ln 3 \)
(d) \( (4x + 1) \cdot 3^{4x+1} \ln 4 \)

2.5.13 \( \frac{d}{dx}(e^x + x^2)^2 = \)
(a) \( 2(e^x + x^2) \)
(b) \( 2(e^x + 2x)(e^x + x^2)^2 \)
(c) \( 2(e^x + x^2)^2 \)
(d) \( 2(e^x + 2x)(e^x + x^2) \)

2.5.14 \( \frac{d}{dx}xe^{-2x} = \)
(a) \( x^2e^{-2x} - 2xe^{-2x} \)
(b) \( 2xe^{-2x} - x^2e^{-2x} \)
(c) \( 2xe^{-2x} - 2xe^{-2x} \)
(d) \( -2xe^{-2x} \)

2.5.15 \( \frac{d}{dx}3^{e^{2x}} = \)
(a) \( 2e^{2x}3^{e^{2x}} \ln 3 \)
(b) \( 2e^{2x}3^{e^{2x}} \)
(c) \( 2 \cdot 3^{e^{2x}} \ln 3 \)
(d) \( e^{2x}3^{e^{2x}} \ln 3 \)
(e) \( 2 \cdot 3^{e^{2x}} \)
2.5. THE CHAIN RULE

2.5.16 If \( \frac{dy}{dx} = 5 \) and \( \frac{dx}{dt} = -2 \) then \( \frac{dy}{dt} = \)

(a) 5
(b) -2
(c) -10
(d) cannot be determined from the information given

2.5.17 If \( \frac{dz}{dx} = 12 \) and \( \frac{dy}{dx} = 2 \) then \( \frac{dz}{dy} = \)

(a) 24
(b) 6
(c) 1/6
(d) cannot be determined from the information given

2.5.18 If \( y = 5x^2 \) and \( \frac{dx}{dt} = 3 \), then when \( x = 4, \frac{dy}{dt} = \)

(a) 12
(b) 80
(c) 120
(d) 15x^2
(e) cannot be determined from the information given

2.5.19 The functions \( f(x) \) and \( h(x) \) are plotted below. The function \( g(x) = f(h(x)) \). What is \( g'(2) \)?

(a) \( g'(2) = -\frac{1}{2} \)
(b) \( g'(2) = 1 \)
(c) \( g'(2) = 3 \)
(d) \( g'(2) = 4 \)
(e) \( g'(2) \) is undefined
2.6 Derivatives of Inverse Functions

Preview Activity 2.6. The equation $y = \frac{5}{9}(x - 32)$ relates a temperature given in $x$ degrees Fahrenheit to the corresponding temperature $y$ measured in degrees Celsius.

(a) Solve the equation $y = \frac{5}{9}(x - 32)$ for $x$ to write $x$ (Fahrenheit temperature) in terms of $y$ (Celsius temperature).

(b) Let $C(x) = \frac{5}{9}(x - 32)$ be the function that takes a Fahrenheit temperature as input and produces the Celsius temperature as output. In addition, let $F(y)$ be the function that converts a temperature given in $y$ degrees Celsius to the temperature $F(y)$ measured in degrees Fahrenheit. Use your work in (a) to write a formula for $F(y)$.

(c) Next consider the new function defined by $p(x) = F(C(x))$. Use the formulas for $F$ and $C$ to determine an expression for $p(x)$ and simplify this expression as much as possible. What do you observe?

(d) Now, let $r(y) = C(F(y))$. Use the formulas for $F$ and $C$ to determine an expression for $r(y)$ and simplify this expression as much as possible. What do you observe?

(e) What is the value of $C'(x)$? of $F'(y)$? How do these values appear to be related?
Activity 2.16.

For each function given below, find its derivative.

(a) \( h(x) = x^2 \ln(x) \)

(b) \( p(t) = \frac{\ln(t)}{e^t + 1} \)

(c) \( s(y) = \ln(\cos(y) + 2) \)

(d) \( z(x) = \tan(\ln(x)) \)

(e) \( m(z) = \ln(\ln(z)) \)
Activity 2.17.

The following prompts in this activity will lead you to develop the derivative of the inverse tangent function.

(a) Let \( r(x) = \arctan(x) \). Use the relationship between the arctangent and tangent functions to rewrite this equation using only the tangent function.

(b) Differentiate both sides of the equation you found in (a). Solve the resulting equation for \( r'(x) \), writing \( r'(x) \) as simply as possible in terms of a trigonometric function evaluated at \( r(x) \).

(c) Recall that \( r(x) = \arctan(x) \). Update your expression for \( r'(x) \) so that it only involves trigonometric functions and the independent variable \( x \).

(d) Introduce a right triangle with angle \( \theta \) so that \( \theta = \arctan(x) \). What are the three sides of the triangle?

(e) In terms of only \( x \) and 1, what is the value of \( \cos(\arctan(x)) \)?

(f) Use the results of your work above to find an expression involving only 1 and \( x \) for \( r'(x) \).
Activity 2.18.

Determine the derivative of each of the following functions.

(a) \( f(x) = x^3 \arctan(x) + e^x \ln(x) \)
(b) \( p(t) = 2^t \arcsin(t) \)
(c) \( h(z) = (\arcsin(5z) + \arctan(4 - z))^{27} \)
(d) \( s(y) = \cot(\arctan(y)) \)
(e) \( m(v) = \ln(\sin^2(v) + 1) \)
(f) \( g(w) = \arctan \left( \frac{\ln(w)}{1 + w^2} \right) \)
Voting Questions

2.6.1 \( \frac{d}{dt} \ln(t^2 + 1) \) is

(a) \( 2t \ln(t^2 + 1) \)
(b) \( \frac{2t}{t^2 + 1} \)
(c) \( \frac{dt}{\ln(t^2 + 1)} \)
(d) \( \frac{1}{t^2 + 1} \)

2.6.2 \( \frac{d}{dx} \ln(1 - x) \) is

(a) \( -\ln(1 - x) \)
(b) \( -2x(1 - x^2)^{-1} \)
(c) \( -(1 - x) \)
(d) \( -(1 - x)^{-1} \)

2.6.3 \( \frac{d}{dx} \ln(\pi) \) is

(a) \( \frac{1}{\pi} \)
(b) \( \frac{\ln(\pi)}{\pi} \)
(c) \( e^\pi \)
(d) 0

2.6.4 \( \frac{d}{d\theta} \ln(\cos \theta) \) is

(a) \( \frac{\sin \theta}{\cos \theta} \)
(b) \( -\sin \theta \ln(\cos \theta) \)
(c) \( \frac{-\sin \theta}{\cos \theta} \)
(d) \( \frac{-\sin \theta}{\ln(\cos \theta)} \)

2.6.5 Find \( f'(x) \) if \( f(x) = \log_5(2x + 1) \).

(a) \( f'(x) = \frac{2}{\ln 5} \cdot \frac{1}{2x + 1} \)
(b) \( f'(x) = \frac{2 \ln 5}{2x + 1} \)

(c) \( f'(x) = \frac{2}{\log_5(2x + 1)} \)

(d) \( f'(x) = \frac{2}{2x + 1} \)

2.6.6 If \( g(x) = \sin^{-1} x \), then \( g'(x) \) is

(a) \( \sqrt{1-x^2} \)

(b) \( \frac{1}{\cos x} \)

(c) \( -\frac{\cos x}{\sin^2 x} \)

(d) \( \csc x \cot x \)

2.6.7 If \( g(x) = (\sin x)^{-1} \), then \( g'(x) \) is

(a) \( \sqrt{1-x^2} \)

(b) \( \frac{1}{\cos x} \)

(c) \( -\frac{\cos x}{\sin^2 x} \)

(d) \( \csc x \cot x \)

2.6.8 If \( p(x) = 3 \ln(2x + 7) \), then \( p'(2) \) is

(a) \( \frac{6}{2x+7} \)

(b) \( \frac{6}{2x+7} \)

(c) \( \frac{3}{2} \)

(d) \( \frac{3}{x} \)

(e) \( \frac{3}{11} \)

2.6.9 If \( q = a^2 \ln(a^3 \sin b + b^2 c) \), then \( \frac{dq}{db} \) is

(a) \( \frac{a^2}{a^3 \sin b + b^2 c} \)

(b) \( \frac{a^3 \cos b + 2a^2 bc}{a^3 \sin b + b^2 c} \)

(c) \( \frac{a^5 \cos b + 2bc}{a^3 \sin b + b^2 c} \)

(d) \( \frac{6a^5 \cos b + 4ab}{a^3 \sin b + b^2 c} \)
2.7 Derivatives of Functions Given Implicitly

Preview Activity 2.7. Let $f$ be a differentiable function of $x$ (whose formula is not known) and recall that $\frac{df}{dx}[f(x)]$ and $f'(x)$ are interchangeable notations. Determine each of the following derivatives of combinations of explicit functions of $x$, the unknown function $f$, and an arbitrary constant $c$.

(a) $\frac{d}{dx}[x^2 + f(x)]$

(b) $\frac{d}{dx}[x^2f(x)]$

(c) $\frac{d}{dx}[c + x + f(x)^2]$

(d) $\frac{d}{dx}[f(x^2)]$

(e) $\frac{d}{dx}[xf(x) + f(cx) + cf(x)]$
Activity 2.19.

Consider the curve defined by the equation \( x = y^5 - 5y^3 + 4y \), whose graph is pictured in Figure 2.4.

Figure 2.4: The curve \( x = y^5 - 5y^3 + 4y \).

(a) Explain why it is not possible to express \( y \) as an explicit function of \( x \).

(b) Use implicit differentiation to find a formula for \( \frac{dy}{dx} \).

(c) Use your result from part (b) to find an equation of the line tangent to the graph of \( x = y^5 - 5y^3 + 4y \) at the point \((0,1)\).

(d) Use your result from part (b) to determine all of the points at which the graph of \( x = y^5 - 5y^3 + 4y \) has a vertical tangent line.
Activity 2.20.

Consider the curve defined by the equation \( y(y^2 - 1)(y - 2) = x(x - 1)(x - 2) \), whose graph is pictured in Figure 2.5. Through implicit differentiation, it can be shown that

\[
\frac{dy}{dx} = \frac{(x - 1)(x - 2) + x(x - 2) + x(x - 1)}{(y^2 - 1)(y - 2) + 2y^2(y - 2) + y(y^2 - 1)}.
\]

Use this fact to answer each of the following questions.

(a) Determine all points \((x, y)\) at which the tangent line to the curve is horizontal.
(b) Determine all points \((x, y)\) at which the tangent line is vertical.
(c) Find the equation of the tangent line to the curve at one of the points where \(x = 1\).
Activity 2.21.

For each of the following curves, use implicit differentiation to find $dy/dx$ and determine the equation of the tangent line at the given point.

(a) $x^3 - y^3 = 6xy$, $(-3, 3)$
(b) $\sin(y) + y = x^3 + x$, $(0, 0)$
(c) $xe^{-xy} = y^2$, $(0.571433, 1)$
Voting Questions

2.7.1 Find $\frac{dy}{dx}$ implicitly if $y^3 = x^2 + 1$.

(a) $\frac{dy}{dx} = \frac{2}{3}x$

(b) $\frac{dy}{dx} = 0$

(c) $\frac{dy}{dx} = \frac{x^2 + 1}{3y^2}$

(d) $\frac{dy}{dx} = \frac{2x}{3y^2}$
2.8 Using Derivatives to Evaluate Limits

Preview Activity 2.8. Let \( h \) be the function given by \( h(x) = \frac{x^5 + x - 2}{x^2 - 1} \).

(a) What is the domain of \( h \)?

(b) Explain why \( \lim_{x \to 1} \frac{x^5 + x - 2}{x^2 - 1} \) results in an indeterminate form.

(c) Next we will investigate the behavior of both the numerator and denominator of \( h \) near the point where \( x = 1 \). Let \( f(x) = x^5 + x - 2 \) and \( g(x) = x^2 - 1 \). Find the local linearizations of \( f \) and \( g \) at \( a = 1 \), and call these functions \( L_f(x) \) and \( L_g(x) \), respectively.

(d) Explain why \( h(x) \approx \frac{L_f(x)}{L_g(x)} \) for \( x \) near \( a = 1 \).

(e) Using your work from (c), evaluate

\[
\lim_{x \to 1} \frac{L_f(x)}{L_g(x)}.
\]

What do you think your result tells us about \( \lim_{x \to 1} h(x) \)?

(f) Investigate the function \( h(x) \) graphically and numerically near \( x = 1 \). What do you think is the value of \( \lim_{x \to 1} h(x) \)?
Activity 2.22.

Evaluate each of the following limits. If you use L’Hopital’s Rule, indicate where it was used, and be certain its hypotheses are met before you apply it.

(a) \( \lim_{x \to 0} \frac{\ln(1 + x)}{x} \)

(b) \( \lim_{x \to \pi} \frac{\cos(x)}{x} \)

(c) \( \lim_{x \to 1} \frac{2 \ln(x)}{1 - e^{x-1}} \)

(d) \( \lim_{x \to 0} \frac{\sin(x) - x}{\cos(2x) - 1} \)
Activity 2.23.

In this activity, we reason graphically to evaluate limits of ratios of functions about which some information is known.

Figure 2.6: Three graphs referenced in the questions of Activity 2.23.

(a) Use the left-hand graph to determine the values of \( f(2) \), \( f'(2) \), \( g(2) \), and \( g'(2) \). Then, evaluate

\[
\lim_{x \to 2} \frac{f(x)}{g(x)}.
\]

(b) Use the middle graph to find \( p(2) \), \( p'(2) \), \( q(2) \), and \( q'(2) \). Then, determine the value of

\[
\lim_{x \to 2} \frac{p(x)}{q(x)}.
\]

(c) Use the right-hand graph to compute \( r(2) \), \( r'(2) \), \( s(2) \), \( s'(2) \). Explain why you cannot determine the exact value of

\[
\lim_{x \to 2} \frac{r(x)}{s(x)}
\]

without further information being provided, but that you can determine the sign of \( \lim_{x \to 2} \frac{r(x)}{s(x)} \). In addition, state what the sign of the limit will be, with justification.
Activity 2.24.

Evaluate each of the following limits. If you use L’Hopital’s Rule, indicate where it was used, and be certain its hypotheses are met before you apply it.

(a) \( \lim_{x \to \infty} \frac{x}{\ln(x)} \)

(b) \( \lim_{x \to \infty} \frac{e^x + x}{2e^x + x^2} \)

(c) \( \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} \)

(d) \( \lim_{x \to \frac{\pi}{2}} \frac{\tan(x)}{x - \frac{\pi}{2}} \)

(e) \( \lim_{x \to \infty} xe^{-x} \)
2.8. USING DERIVATIVES TO EVALUATE LIMITS

Voting Questions

2.8.1 Evaluate \( \lim_{x \to \pi/2^-} \frac{1 + \tan x}{\sec x} \).

(a) 0
(b) 1
(c) \( \infty \)
(d) \( -\infty \)

Other Voting Questions for Derivatives

2.8.2 A function intersects the \( x \)-axis at points \( a \) and \( b \), where \( a < b \). The slope of the function at \( a \) is positive and the slope at \( b \) is negative. Which of the following is true for any such function? There exists some point on the interval \((a, b)\) where

(a) the slope is zero and the function has a local maximum.
(b) the slope is zero but there is not a local maximum.
(c) there is a local maximum, but there doesn’t have to be a point at which the slope is zero.
(d) none of the above have to be true.

2.8.3 A function intersects the \( x \)-axis at points \( a \) and \( b \), where \( a < b \). The slope of the function at \( a \) is positive and the slope at \( b \) is negative. Which of the following is true for any such function for which the limit of the function exists and is finite at every point? There exists some point on the interval \((a, b)\) where

(a) the slope is zero and the function has a local maximum.
(b) the slope is zero but there is not a local maximum.
(c) there is a local maximum, but there doesn’t have to be a point at which the slope is zero.
(d) none of the above have to be true.

2.8.4 A continuous function intersects the \( x \)-axis at points \( a \) and \( b \), where \( a < b \). The slope of the function at \( a \) is positive and the slope at \( b \) is negative. Which of the following is true for any such function? There exists some point on the interval \((a, b)\) where

(a) the slope is zero and the function has a local maximum.
(b) the slope is zero but there is not a local maximum.
(c) there is a local maximum, but there doesn't have to be a point at which the slope is zero.
(d) none of the above have to be true.

2.8.5 A continuous and differentiable function intersects the x-axis at points \( a \) and \( b \), where \( a < b \). The slope of the function at \( a \) is positive and the slope at \( b \) is negative. Which of the following is true for any such function? There exists some point on the interval \((a, b)\) where

(a) the slope is zero and the function has a local maximum.
(b) the slope is zero but there is not a local maximum.
(c) there is a local maximum, but there doesn't have to be a point at which the slope is zero.
(d) none of the above have to be true.

2.8.6 On a toll road a driver takes a time stamped toll-card from the starting booth and drives directly to the end of the toll section. After paying the required toll, the driver is surprised to receive a speeding ticket along with the toll receipt. Which of the following best describes the situation?

(a) The booth attendant does not have enough information to prove that the driver was speeding.
(b) The booth attendant can prove that the driver was speeding during his trip.
(c) The driver will get a ticket for a lower speed than his actual maximum speed.
(d) Both (b) and (c).

2.8.7 True or False: For \( f(x) = |x| \) on the interval \([-\frac{1}{2}, 2]\), you can find a point \( c \) in \((-\frac{1}{2}, 2)\), such that \( f'(c) = \frac{f(2) - f(-\frac{1}{2})}{2 - (-\frac{1}{2})} \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.8.8 A racer is running back and forth along a straight path. He finishes the race at the place where he began.

True or False: There had to be at least one moment, other than the beginning and the end of the race, when he “stopped” (i.e., his speed was 0).
2.8. USING DERIVATIVES TO EVALUATE LIMITS

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.8.9 Two racers start a race at the same moment and finish in a tie. Which of the following must be true?

(a) At some point during the race the two racers were not tied.
(b) The racers’ speeds at the end of the race must have been exactly the same.
(c) The racers must have had the same speed at exactly the same time at some point in the race.
(d) The racers had to have the same speed at some moment, but not necessarily at exactly the same time.

2.8.10 Two horses start a race at the same time and one runs slower than the other throughout the race. True or False: The Racetrack Principle can be used to justify the fact that the slower horse loses the race.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.8.11 True or False: The Racetrack Principle can be used to justify the statement that if two horses start a race at the same time, the horse that wins must have been moving faster than the other throughout the race.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.8.12 Which of the following statements illustrates a correct use of the Racetrack Principle?

(a) Since $\sin 0 = 0$ and $\cos x \leq 1$ for all $x$, the Racetrack Principle tells us that $\sin x \leq x$ for all $x \geq 0$. 
(b) For $a < b$, if $f'(x)$ is positive on $[a, b]$ then the Racetrack Principle tells us that $f(a) < f(b)$.

(c) Let $f(x) = x$ and $g(x) = x^2 - 2$. Since $f(-1) = g(-1) = -1$ and $f(1) > g(1)$, the Racetrack Principle tells us that $f'(x) > g'(x)$ for $-1 < x < 1$.

(d) All are correct uses of the Racetrack Principle.

(e) Exactly 2 of a, b, and c are correct uses of the Racetrack Principle.
2.9 More on Limits

Preview Activity 2.9. Evaluate the following limits.

(a) \( \lim_{x \to -1} \frac{x^2 - x - 2}{x^3 - x} \)

(b) \( \lim_{x \to 1} \frac{x^2 - x - 2}{x^3 - x} \)

(c) \( \lim_{x \to 2} \arcsin(x^2 - 5) \)

(d) \( \lim_{x \to 0} \frac{\sin 3x}{4x} \)

(e) \( \lim_{x \to 2} \frac{|x^2 + x - 6|}{x - 2} \)

(f) \( \lim_{x \to 0^+} x \ln x^4 \)
Preview Activity 2.10. Write an equivalent expression for each expression below by multiplying and dividing by the conjugate. Simplify your answer as much as possible.

(a) \( \frac{\sqrt{x} - 1}{x - 1} \)

(b) \( \frac{x - 16}{\sqrt{x} - 4} \)

(c) \( \frac{x}{\sqrt{7 + x} - \sqrt{7}} \)

(d) \( \sqrt{9x^2 + x - 3x} \)

Use your work to evaluate the following limits.

(a) \( \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \)

(b) \( \lim_{x \to 16} \frac{x - 16}{\sqrt{x} - 4} \)

(c) \( \lim_{x \to 0} \frac{x}{\sqrt{7 + x} - \sqrt{7}} \)

(d) \( \lim_{x \to \infty} \sqrt{9x^2 + x - 3x} \)
Activity 2.25.

In this exercise we wish to graph the curve \( y = \sqrt{4x^2 - 1} \).

(a) Find the domain of the function \( f(x) = \sqrt{4x^2 - 1} \). Notice that some negative values of \( x \) are allowed in the domain.

(b) Evaluate the derivative of \( f(x) \) and find the intervals where the function is increasing and decreasing.

(c) Evaluate the limit \( \lim_{x \to \infty} \sqrt{4x^2 - 1} - 2x \).

(d) If \( y = ax + b \) is a slant asymptote of \( y = f(x) \), then the values of \( ax + b \) and \( f(x) \) are getting closer for increasingly large values of \( x \) (we only discuss the case were \( x \to \infty \), but the case where \( x \to -\infty \) is similar). In other words, \( y = ax + b \) is a slant asymptote of \( y = f(x) \) if

\[
\lim_{x \to \infty} f(x) - (ax + b) = 0.
\]

Use the limit in (c) to find a slant asymptote for the graph of \( y = f(x) \).

![Figure 2.7: This figure illustrates a function \( f \) which has a slant asymptote \( y = ax + b \).](image)

(e) When \( x \) is large and positive, \( 4x^2 \) is much larger than 1, and we may write \( \sqrt{4x^2 - 1} \approx \sqrt{4x^2} \). Simplify \( \sqrt{4x^2} \) to obtain another argument showing that \( y = 2x \) is a slant asymptote of the graph of \( y = f(x) \). How would this argument work out for large but negative values of \( x \)? Hint: \( \sqrt{4x^2} \) is not equal to \( 2x \) when \( x \) is negative. What is it equal to?

(f) Graph \( y = f(x) \) for nonnegative values of \( x \) using information on the domain, the derivative and the asymptote.

(g) Show \( f(-1) = f(1), f(-2.4) = f(2.4) \), and in general that \( f(-x) = f(x) \) to argue that \( f \) is symmetrical with respect to the \( y \)-axis. Use your previous graph to obtain the graph of \( f \) over its domain.

(h) Consider the curve \( y = \sqrt{9x^2 + x - 1} + 1 \) and find its slant asymptotes.
Activity 2.26.

Evaluate the following limits.

(a) \( \lim_{x \to 2} \cos \left( \frac{\pi (x^2 - 4)}{x - 2} \right) \)

(b) \( \lim_{x \to \infty} \sqrt{\frac{2x^2 + x - 1}{3x^2 - 4}} \)

(c) \( \lim_{x \to 9} e^{\frac{\sqrt{x} - 3}{x}} \)

(d) \( \lim_{x \to \infty} \arcsin \left( \frac{\sqrt{3} + x}{2x} \right) \)
Activity 2.27.

Find all values where the following functions are continuous.

(a) $\sqrt{1-x^2}$

(b) $x^2 + 4 + \frac{1}{\sqrt{1-x^2}}$

(c) $\cos\left(e^{1/x}\right)$

(d) $\arctan\left(\frac{x^2 - 4}{3x^2 - 6x}\right)$
Activity 2.28.

Evaluate the following limits

(a) \( \lim_{x \to 0^+} \sqrt{x} \sin \left( \frac{1}{x^2} \right) \)

(b) \( \lim_{x \to \infty} \sqrt{x} \arctan \left( \frac{1}{x} \right) \) (hint: \( 0 \leq \arctan z \leq z \) for \( z \geq 0 \) and notice that when \( x \) is positive, \( 1/x \) is also positive.)

(c) Define for an integer \( n \) the factorial of \( n \) written as \( n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n - 1) \times n \). For instance, \( 4! = 1 \times 2 \times 3 \times 4 = 24 \). Use the Squeeze Theorem to evaluate the following limit

\( \lim_{n \to \infty} \frac{2^n}{n!} \).

Here are a few steps you may want to follow:

1. Consider \( n \) as a large positive integer and argue that

\( \frac{2^n}{n!} = \left( \frac{2}{1} \right) \cdot \left( \frac{2}{2} \right) \cdot \left( \frac{2}{3} \right) \cdot \left( \frac{2}{4} \right) \ldots \left( \frac{2}{n} \right) \).

2. Use the previous equality and the fact that \( 2/i \leq 2/3 \) for \( i \geq 3 \) to show that

\( 0 \leq \frac{2^n}{n!} \leq 2 \cdot \left( \frac{2}{3} \right)^{n-2} \).

3. Use the Squeeze Theorem to find

\( \lim_{n \to \infty} \frac{2^n}{n!} \).

4. Which function dominates as \( n \to \infty \): the exponential function \( 2^n \) or the factorial function \( n! \)?

(d) In Appendix B, read the proof for the differentiation formula

\( \frac{d}{dx} [\sin(x)] = \cos(x) \)

and identify where the Squeeze Theorem is used.

\( \triangledown \)
Chapter 3

Using Derivatives

3.1 Using derivatives to identify extreme values of a function

Preview Activity 3.1. Consider the function \( h \) given by the graph in Figure 3.1. Use the graph to answer each of the following questions.

(a) Identify all of the values of \( c \) for which \( h(c) \) is a local maximum of \( h \).

(b) Identify all of the values of \( c \) for which \( h(c) \) is a local minimum of \( h \).

(c) Does \( h \) have a global maximum? If so, what is the value of this global maximum?

(d) Does \( h \) have a global minimum? If so, what is its value?

(e) Identify all values of \( c \) for which \( h'(c) = 0 \).

Figure 3.1: The graph of a function \( h \) on the interval \([-3, 3]\).
(f) Identify all values of $c$ for which $h'(c)$ does not exist.

(g) True or false: every relative maximum and minimum of $h$ occurs at a point where $h'(c)$ is either zero or does not exist.

(h) True or false: at every point where $h'(c)$ is zero or does not exist, $h$ has a relative maximum or minimum.
Activity 3.1.

Suppose that \( g(x) \) is a function continuous for every value of \( x \neq 2 \) whose first derivative is
\[
g'(x) = \frac{(x + 4)(x - 1)^2}{x - 2}.
\]
Further, assume that it is known that \( g \) has a vertical asymptote at \( x = 2 \).

(a) Determine all critical values of \( g \).

(b) By developing a carefully labeled first derivative sign chart, decide whether \( g \) has as a local maximum, local minimum, or neither at each critical value.

(c) Does \( g \) have a global maximum? global minimum? Justify your claims.

(d) What is the value of \( \lim_{x \to \infty} g'(x) \)? What does the value of this limit tell you about the long-term behavior of \( g \)?

(e) Sketch a possible graph of \( y = g(x) \).
Activity 3.2.
Suppose that $g$ is a function whose second derivative, $g''$, is given by the following graph.

![Graph of $g''(x)$](image)

Figure 3.2: The graph of $y = g''(x)$.

(a) Find all points of inflection of $g$.

(b) Fully describe the concavity of $g$ by making an appropriate sign chart.

(c) Suppose you are given that $g'(-1.67857351) = 0$. Is there a local maximum, local minimum, or neither (for the function $g$) at this critical value of $g$, or is it impossible to say? Why?

(d) Assuming that $g''(x)$ is a polynomial (and that all important behavior of $g''$ is seen in the graph above, what degree polynomial do you think $g(x)$ is? Why?

\[\triangledown\]
Activity 3.3.

Consider the family of functions given by \( h(x) = x^2 + \cos(kx) \), where \( k \) is an arbitrary positive real number.

(a) Use a graphing utility to sketch the graph of \( h \) for several different \( k \)-values, including \( k = 1, 3, 5, 10 \). Plot \( h(x) = x^2 + \cos(3x) \) on the axes provided below. What is the smallest value of \( k \) at which you think you can see (just by looking at the graph) at least one inflection point on the graph of \( h \)?

(b) Explain why the graph of \( h \) has no inflection points if \( k \leq \sqrt{2} \), but infinitely many inflection points if \( k > \sqrt{2} \).

(c) Explain why, no matter the value of \( k \), \( h \) can only have a finite number of critical values.

\[\text{Figure 3.3: Axes for plotting } y = h(x).\]
3.1. USING DERIVATIVES TO IDENTIFY EXTREME VALUES OF A FUNCTION

Voting Questions

3.1.1 If \( f''(a) = 0 \), then \( f \) has an inflection point at \( a \).
   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

3.1.2 A local maximum of \( f \) only occurs at a point where \( f'(x) = 0 \).
   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

3.1.3 If \( x = p \) is not a local minimum or maximum of \( f \), then \( x = p \) is not a critical point of \( f \).
   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

3.1.4 If \( f'(x) \) is continuous and \( f(x) \) has no critical points, then \( f \) is everywhere increasing or everywhere decreasing.
   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

3.1.5 If \( f'(x) \geq 0 \) for all \( x \), then \( f(a) \leq f(b) \) whenever \( a \leq b \).
   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
(d) False, and I am very confident

3.1.6 Imagine that you are skydiving. The graph of your speed as a function of time from the time you jumped out of the plane to the time you achieved terminal velocity is

(a) increasing and concave up
(b) decreasing and concave up
(c) increasing and concave down
(d) decreasing and concave down

3.1.7 Water is being poured into a “Dixie cup” (a standard cup that is smaller at the bottom than at the top). The height of the water in the cup is a function of the volume of water in the cup. The graph of this function is

(a) increasing and concave up
(b) increasing and concave down
(c) a straight line with positive slope.
3.2 Using derivatives to describe families of functions

Preview Activity 3.2. Let \( a, h, \) and \( k \) be arbitrary real numbers with \( a \neq 0 \), and let \( f \) be the function given by the rule \( f(x) = a(x - h)^2 + k \). See http://www.geogebratube.org/student/m68215 to explore the effect that \( a, h, \) and \( k \) have on this function.

(a) What familiar type of function is \( f \)? What information do you know about \( f \) just by looking at its form? (Think about the roles of \( a, h, \) and \( k \).)

(b) Next we use some calculus to develop familiar ideas from a different perspective. To start, treat \( a, h, \) and \( k \) as constants and compute \( f'(x) \).

(c) Find all critical values of \( f \). (These will depend on at least one of \( a, h, \) and \( k \).)

(d) Assume that \( a < 0 \). Construct a first derivative sign chart for \( f \).

(e) Based on the information you’ve found above, classify the critical values of \( f \) as maxima or minima.
Activity 3.4.

Consider the family of functions defined by \( p(x) = x^3 - ax \), where \( a \neq 0 \) is an arbitrary constant. See the geogebra applet http://www.geogebratube.org/student/m110845.

(a) Find \( p'(x) \) and determine the critical values of \( p \). How many critical values does \( p \) have?

(b) Construct a first derivative sign chart for \( p \). What can you say about the overall behavior of \( p \) if the constant \( a \) is positive? Why? What if the constant \( a \) is negative? In each case, describe the relative extremes of \( p \).

(c) Find \( p''(x) \) and construct a second derivative sign chart for \( p \). What does this tell you about the concavity of \( p \)? What role does \( a \) play in determining the concavity of \( p \)?

(d) Without using a graphing utility, sketch and label typical graphs of \( p(x) \) for the cases where \( a > 0 \) and \( a < 0 \). Label all inflection points and local extrema.

(e) Finally, use a graphing utility to test your observations above by entering and plotting the function \( p(x) = x^3 - ax \) for at least four different values of \( a \). Write several sentences to describe your overall conclusions about how the behavior of \( p \) depends on \( a \).
Activity 3.5.

Consider the two-parameter family of functions of the form \( h(x) = a(1 - e^{-bx}) \), where \( a \) and \( b \) are positive real numbers.

(a) Find the first derivative and the critical values of \( h \). Use these to construct a first derivative sign chart and determine for which values of \( x \) the function \( h \) is increasing and decreasing.

(b) Find the second derivative and build a second derivative sign chart. For which values of \( x \) is a function in this family concave up? concave down?

(c) What is the value of \( \lim_{x \to \infty} a(1 - e^{-bx}) \)? \( \lim_{x \to -\infty} a(1 - e^{-bx}) \)?

(d) How does changing the value of \( b \) affect the shape of the curve?

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function \( h \) and how this behavior depends on \( a \) and \( b \).
Activity 3.6.

Let \( L(t) = \frac{A}{1 + ce^{-kt}} \), where \( A, c, \) and \( k \) are all positive real numbers.

(a) Observe that we can equivalently write \( L(t) = A(1 + ce^{-kt})^{-1} \). Find \( L'(t) \) and explain why \( L \) has no critical values. Is \( L \) always increasing or always decreasing? Why?

(b) Given the fact that
\[
L''(t) = Ack^2e^{-kt} \frac{ce^{-kt} - 1}{(1 + ce^{-kt})^3},
\]
find all values of \( t \) such that \( L''(t) = 0 \) and hence construct a second derivative sign chart. For which values of \( t \) is a function in this family concave up? concave down?

(c) What is the value of \( \lim_{t \to \infty} \frac{A}{1 + ce^{-kt}} \)? \( \lim_{t \to -\infty} \frac{A}{1 + ce^{-kt}} \)?

(d) Find the value of \( L(x) \) at the inflection point found in (b).

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function \( h \) and how this behavior depends on \( a \) and \( b \).

(f) Explain why it is reasonable to think that the function \( L(t) \) models the growth of a population over time in a setting where the largest possible population the surrounding environment can support is \( A \).
Voting Questions

3.2.1 The functions in the figure have the form \( y = A \sin x \). Which of the functions has the largest \( A \)? Assume the scale on the vertical axes is the same for each graph.

3.2.2 The functions in the figure have the form \( y = \sin(Bx) \). Which of the functions has the largest \( B \)? Assume the scale on the horizontal axes is the same for each graph.

3.2.3 Let \( f(x) = ax + \frac{b}{x} \). What are the critical points of \( f(x) \)?
   (a) \(-b/a\)  (b) 0  (c) \(\pm \sqrt{b/a}\)  (d) \(\pm \sqrt{-b/a}\)  (e) No critical points

3.2.4 Let \( f(x) = ax + \frac{b}{x} \). Suppose \( a \) and \( b \) are positive. What happens to \( f(x) \) as \( b \) increases?
   (a) The critical points move further apart.
   (b) The critical points move closer together.

3.2.5 Let \( f(x) = ax + \frac{b}{x} \). Suppose \( a \) and \( b \) are positive. What happens to \( f(x) \) as \( b \) increases?
   (a) The critical values move further apart.
   (b) The critical values move closer together.
3.2.6 Let $f(x) = ax + b/x$. Suppose $a$ and $b$ are positive. What happens to $f(x)$ as $a$ increases?

(a) The critical points move further apart.
(b) The critical points move closer together.

3.2.7 Let $f(x) = ax + b/x$. Suppose $a$ and $b$ are positive. What happens to $f(x)$ as $a$ increases?

(a) The critical values move further apart.
(b) The critical values move closer together.

3.2.8 Find a formula for a parabola with its vertex at (3,2) and with a second derivative of -4.

(a) $y = -4x^2 + 48x - 106$.
(b) $y = -4x^2 + 24x - 34$.
(c) $y = -2x^2 + 12x - 16$.
(d) $y = -2x^2 + 4x + 8$. 
3.3 Global Optimization

Preview Activity 3.3. Let \( f(x) = 2 + \frac{3}{1 + (x + 1)^2} \).

(a) Determine all of the critical values of \( f \).

(b) Construct a first derivative sign chart for \( f \) and thus determine all intervals on which \( f \) is increasing or decreasing.

(c) Does \( f \) have a global maximum? If so, why, and what is its value and where is the maximum attained? If not, explain why.

(d) Determine \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

(e) Explain why \( f(x) > 2 \) for every value of \( x \).

(f) Does \( f \) have a global minimum? If so, why, and what is its value and where is the minimum attained? If not, explain why.
Activity 3.7.

Let \( g(x) = \frac{1}{3}x^3 - 2x + 2 \).

(a) Find all critical values of \( g \) that lie in the interval \(-2 \leq x \leq 3\).

(b) Use a graphing utility to construct the graph of \( g \) on the interval \(-2 \leq x \leq 3\).

(c) From the graph, determine the \( x \)-values at which the global minimum and global maximum of \( g \) occur on the interval \([-2, 3]\).

(d) How do your answers change if we instead consider the interval \(-2 \leq x \leq 2\)?

(e) What if we instead consider the interval \(-2 \leq x \leq 1\)?
Activity 3.8.

Find the exact absolute maximum and minimum of each function on the stated interval.

(a) \( h(x) = xe^{-x} \), \([0, 3]\)

(b) \( p(t) = \sin(t) + \cos(t) \), \([-\frac{\pi}{2}, \frac{\pi}{2}]\)

(c) \( q(x) = \frac{x^2}{x-2} \), \([3, 7]\)

(d) \( f(x) = 4 - e^{-(x-2)^2} \), \((-\infty, \infty)\)
Activity 3.9.

A piece of cardboard that is $10 \times 15$ (each measured in inches) is being made into a box without a top. To do so, squares are cut from each corner of the box and the remaining sides are folded up. If the box needs to be at least 1 inch deep and no more than 3 inches deep, what is the maximum possible volume of the box? what is the minimum volume? Justify your answers using calculus.

(a) Draw a labeled diagram that shows the given information. What variable should we introduce to represent the choice we make in creating the box? Label the diagram appropriately with the variable, and write a sentence to state what the variable represents.

(GeoGebra applet for the box folding problem)

(b) Determine a formula for the function $V$ (that depends on the variable in (a)) that tells us the volume of the box.

(c) What is the domain of the function $V$? That is, what values of $x$ make sense for input? Are there additional restrictions provided in the problem?

(d) Determine all critical values of the function $V$.

(e) Evaluate $V$ at each of the endpoints of the domain and at any critical values that lie in the domain.

(f) What is the maximum possible volume of the box? the minimum?
Voting Questions

3.3.1 If \( f(x) \) is continuous on a closed interval, then it is enough to look at the points where \( f'(x) = 0 \) in order to find its global maxima and minima.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.3.2 A function defined on all points of a closed interval always has a global maximum and a global minimum.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.3.3 Let \( f \) be a continuous function on the closed interval \( 0 \leq x \leq 1 \). There exists a positive number \( A \) so that the graph of \( f \) can be drawn inside the rectangle \( 0 \leq x \leq 1, -A \leq y \leq A \).

The above statement is:

(a) Always true.
(b) Sometimes true.
(c) Not enough information.

3.3.4 Let \( f(x) = x^2 \). \( f \) has an upper bound on the interval \( (0, 2) \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.3.5 Let \( f(x) = x^2 \). \( f \) has a global maximum on the interval \( (0, 2) \).

(a) True, and I am very confident
3.3.6 Let \( f(x) = x^2 \). \( f \) has a global minimum on the interval \((0, 2)\).

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident

3.3.7 Let \( f(x) = x^2 \). \( f \) has a global minimum on any interval \([a, b]\).

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident

3.3.8 Consider \( f(x) = -3x^2 + 12x + 7 \) on the interval \(-2 \leq x \leq 4\). Where does this function have its global maximum value?

(a) \( x = -2 \)  
(b) \( x = 0 \)  
(c) \( x = 2 \)  
(d) \( x = 4 \)

3.3.9 Consider \( f(x) = -3x^2 + 12x + 7 \) on the interval \(-2 \leq x \leq 4\). Where does this function have its global minimum value?

(a) \( x = -2 \)  
(b) \( x = 0 \)  
(c) \( x = 2 \)  
(d) \( x = 4 \)
3.4 Introduction to Sensitivity Analysis

Preview Activity 3.4. Consider the functions below. Each is defined with parameters that may not be known exactly. Find an expression for the critical point(s) for each function.

(a) \( f(x) = x^3 - ax \)

(b) \( f(N) = \frac{N}{1 + kN^2} \)

(c) \( f(z) = ze^{-z^2/\sigma} \)

(d) \( f(t) = a(1 - e^{-bt}) \)
Activity 3.10.

Consider the family of functions defined by

\[ f(x) = \frac{x}{1+kx^2} \]

where \( k > 0 \) is an arbitrary constant. The first and second derivatives are

\[ f'(x) = \frac{1 - kx^2}{(1+kx^2)^2} \quad \text{and} \quad f''(x) = \frac{2kx(kx^2 - 3)}{(1+kx^2)^3} \]

(a) Determine the critical values of \( f \) in terms of \( k \). How many critical values does \( f \) have?

(b) Construct a first derivative sign chart for \( f \) and discuss the behavior of the critical points.

(c) From part (a) you should have found two critical values. Write these values as functions of \( k \) and find the derivatives with respect to \( k \).

(d) Based on your answers to part (c) describe the sensitivity of the critical values of \( f \) with respect to the parameter \( k \). Create plots for critical values vs. \( k \) and the derivatives of the critical values vs. \( k \).
3.4. INTRODUCTION TO SENSITIVITY ANALYSIS

Voting Questions

3.4.1 What are the critical points of the function \( f(x) = x^3 - ax \)?

(a) \( x = 0, \pm \sqrt{a} \)  
(b) \( x = \pm \sqrt{a/3} \)  
(c) \( x = \pm \sqrt{3a} \)  
(d) \( x = \pm \sqrt{-a/3} \)  
(e) \( x = \pm \sqrt{-3a} \)  
(f) No critical points exist.

3.4.2 Consider the critical points of the function \( f(x) = x^3 - ax \). Are these critical points more sensitive to large values of \( a \) or to small values of \( a \)?

(a) Slight changes to large values of \( a \) affect the critical points more.  
(b) Slight changes to small values of \( a \) affect the critical points more.  
(c) Slight changes to \( a \) affect the critical points equally, whether \( a \) is large or small.

3.4.3 We have a system where the critical points are as follows: \( x = \pm 5A^2/B \). Will the critical points be more sensitive to changes in \( A \) when \( A \) is small, or when \( A \) is large?

(a) The critical points will be more sensitive to changes in \( A \) when \( A \) is small.  
(b) The critical points will be more sensitive to changes in \( A \) when \( A \) is large.  
(c) The critical points will be equally affected by changes in \( A \), no matter whether \( A \) is small or large.

3.4.4 What are the critical points of the function \( f(z) = ze^{-z^2/\sigma} \)?

(a) \( z = 0 \)  
(b) \( z = \pm \sqrt{2} \)  
(c) \( z = \pm \sqrt{1/2} \)  
(d) \( z = \pm \sqrt{\sigma} \)  
(e) \( z = \frac{\sigma}{2} \)  
(f) No critical points exist.
3.4.5 Consider the critical points of the function \( f(z) = ze^{-z^2/\sigma} \). Are these critical points more sensitive to large or small values of \( \sigma \)?

(a) The critical points are more sensitive to small values of \( \sigma \).
(b) The critical points are more sensitive to large values of \( \sigma \).
(c) The critical points are equally sensitive to large and small values of \( \sigma \).

3.4.6 What are the critical points of the function \( f(N) = \frac{N}{1+KN^2} \)?

(a) \( N = \sqrt{\frac{1}{k}} \)
(b) \( N = \sqrt{k} \)
(c) \( N = \pm \sqrt{\frac{1}{k}} \)
(d) \( N = \pm \sqrt{\frac{1}{2k}} \)
(e) No critical points exist.

3.4.7 Consider the critical points of the function \( f(N) = \frac{N}{1+KN^2} \). Will the critical points be more sensitive to changes in \( k \) when \( k \) is large or when \( k \) is small?

(a) Critical points will be more sensitive to changes in \( k \) when \( k \) is large.
(b) Critical points will be more sensitive to changes in \( k \) when \( k \) is small.
(c) Critical points will be equally affected by changes in \( k \) no matter whether \( k \) is small or large.

3.4.8 What are the critical points of the function \( f(t) = a(1 - e^{-bt}) \)?

(a) \( t = 0 \)
(b) \( t = \pm \sqrt{\frac{b}{a}} \)
(c) \( t = -\frac{1}{b} \)
(d) \( t = -\frac{1}{b} \ln \frac{1}{ab} \)
(e) No critical points exist.
3.5 Applied Optimization

Preview Activity 3.5. According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where by “girth” we mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?

(a) Let \( x \) represent the length of one side of the square end and \( y \) the length of the longer side. Label these quantities appropriately on the image shown in Figure 3.4.

(b) What is the quantity to be optimized in this problem? Find a formula for this quantity in terms of \( x \) and \( y \).

(c) The problem statement tells us that the parcel’s girth plus length may not exceed 108 inches. In order to maximize volume, we assume that we will actually need the girth plus length to equal 108 inches. What equation does this produce involving \( x \) and \( y \)?

(d) Solve the equation you found in (c) for one of \( x \) or \( y \) (whichever is easier).

(e) Now use your work in (b) and (d) to determine a formula for the volume of the parcel so that this formula is a function of a single variable.

(f) Over what domain should we consider this function? Note that both \( x \) and \( y \) must be positive; how does the constraint that girth plus length is 108 inches produce intervals of possible values for \( x \) and \( y \)?

(g) Find the absolute maximum of the volume of the parcel on the domain you established in (f) and hence also determine the dimensions of the box of greatest volume. Justify that you’ve found the maximum using calculus.
Activity 3.11.

A soup can in the shape of a right circular cylinder is to be made from two materials. The material for the side of the can costs $0.015 per square inch and the material for the lids costs $0.027 per square inch. Suppose that we desire to construct a can that has a volume of 16 cubic inches. What dimensions minimize the cost of the can?

(a) Draw a picture of the can and label its dimensions with appropriate variables.

(b) Use your variables to determine expressions for the volume, surface area, and cost of the can.

(c) Determine the total cost function as a function of a single variable. What is the domain on which you should consider this function?

(d) Find the absolute minimum cost and the dimensions that produce this value.

(e) How sensitive is your solution in part (d) to the cost of the lid? In other words, if the cost of the lid were to change by a small amount would that cause large or small changes in the minimum cost?
Activity 3.12.

A hiker starting at a point $P$ on a straight road walks east towards point $Q$, which is on the road and 3 kilometers from point $P$. Two kilometers due north of point $Q$ is a cabin. The hiker will walk down the road for a while, at a pace of 8 kilometers per hour. At some point $Z$ between $P$ and $Q$, the hiker leaves the road and makes a straight line towards the cabin through the woods, hiking at a pace of 3 kph, as pictured in Figure 3.5. In order to minimize the time to go from $P$ to $Z$ to the cabin, where should the hiker turn into the forest? How sensitive is your solution on the hiker’s pace through the woods?

![Figure 3.5: A hiker walks from $P$ to $Z$ to the cabin, as pictured.](image)
Activity 3.13.

(a) Consider the region in the $x$-$y$ plane that is bounded by the $x$-axis and the function $f(x) = 25 - x^2$. Construct a rectangle whose base lies on the $x$-axis and is centered at the origin, and whose sides extend vertically until they intersect the curve $y = 25 - x^2$. Which such rectangle has the maximum possible area? Which such rectangle has the greatest perimeter? Which has the greatest combined perimeter and area?

(b) Answer the same questions in terms of positive parameters $a$ and $b$ for the function $f(x) = b - ax^2$. 

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Activity 3.14.

A trough is being constructed by bending a $4 \times 24$ (measured in feet) rectangular piece of sheet metal. Two symmetric folds 2 feet apart will be made parallel to the longest side of the rectangle so that the trough has cross-sections in the shape of a trapezoid, as pictured in Figure 3.6. At what angle should the folds be made to produce the trough of maximum volume? How sensitive is your solution to the 2 foot distance between the folds?

Figure 3.6: A cross-section of the trough formed by folding to an angle of $\theta$. 
Voting Questions

3.5.1 (no clicker questions for this section)
3.6 Related Rates

**Preview Activity 3.6.** A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon’s diameter is 12 inches? Is the radius changing more rapidly when \( d = 12 \) or when \( d = 16 \)? Why?

(a) Draw several spheres with different radii, and observe that as volume changes, the radius, diameter, and surface area of the balloon also change.

(b) Recall that the volume of a sphere of radius \( r \) is \( V = \frac{4}{3}\pi r^3 \). Note well that in the setting of this problem, both \( V \) and \( r \) are changing as time \( t \) changes, and thus both \( V \) and \( r \) may be viewed as implicit functions of \( t \), with respective derivatives \( \frac{dV}{dt} \) and \( \frac{dr}{dt} \).

Differentiate both sides of the equation \( V = \frac{4}{3}\pi r^3 \) with respect to \( t \) (using the chain rule on the right) to find a formula for \( \frac{dV}{dt} \) that depends on both \( r \) and \( \frac{dr}{dt} \).

(c) At this point in the problem, by differentiating we have “related the rates” of change of \( V \) and \( r \). Recall that we are given in the problem that the balloon is being inflated at a constant rate of 20 cubic inches per second. Is this rate the value of \( \frac{dr}{dt} \) or \( \frac{dV}{dt} \)? Why?

(d) From part (c), we know the value of \( \frac{dV}{dt} \) at every value of \( t \). Next, observe that when the diameter of the balloon is 12, we know the value of the radius. In the equation \( \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \), substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is \( \frac{dr}{dt} \). How fast is the radius changing at the instant \( d = 12 \)?

(e) How is the situation different when \( d = 16 \)? When is the radius changing more rapidly, when \( d = 12 \) or when \( d = 16 \)?
Activity 3.15.

A water tank has the shape of an inverted circular cone (point down) with a base of radius 6 feet and a depth of 8 feet. Suppose that water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute.

(a) Draw a picture of the conical tank, including a sketch of the water level at a point in time when the tank is not yet full. Introduce variables that measure the radius of the water’s surface and the water’s depth in the tank, and label them on your figure.

(b) Say that $r$ is the radius and $h$ the depth of the water at a given time, $t$. What equation relates the radius and height of the water, and why?

(c) Determine an equation that relates the volume of water in the tank at time $t$ to the depth $h$ of the water at that time.

(d) Through differentiation, find an equation that relates the instantaneous rate of change of water volume with respect to time to the instantaneous rate of change of water depth at time $t$.

(e) Find the instantaneous rate at which the water level is rising when the water in the tank is 3 feet deep.

(f) When is the water rising most rapidly: at $h = 3$, $h = 4$, or $h = 5$?
Activity 3.16.

A television camera is positioned 4000 feet from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. In addition, the auto-focus of the camera has to take into account the increasing distance between the camera and the rocket. We assume that the rocket rises vertically. (A similar problem is discussed and pictured dynamically at http://gvsu.edu/s/9t. Exploring the applet at the link will be helpful to you in answering the questions that follow.)

(a) Draw a figure that summarizes the given situation. What parts of the picture are changing? What parts are constant? Introduce appropriate variables to represent the quantities that are changing.

(b) Find an equation that relates the camera’s angle of elevation to the height of the rocket, and then find an equation that relates the instantaneous rate of change of the camera’s elevation angle to the instantaneous rate of change of the rocket’s height (where all rates of change are with respect to time).

(c) Find an equation that relates the distance from the camera to the rocket to the rocket’s height, as well as an equation that relates the instantaneous rate of change of distance from the camera to the rocket to the instantaneous rate of change of the rocket’s height (where all rates of change are with respect to time).

(d) Suppose that the rocket’s speed is 600 ft/sec at the instant it has risen 3000 feet. How fast is the distance from the television camera to the rocket changing at that moment? If the camera is following the rocket, how fast is the camera’s angle of elevation changing at that same moment?

(e) If from an elevation of 3000 feet onward the rocket continues to rise at 600 feet/sec, will the rate of change of distance with respect to time be greater when the elevation is 4000 feet than it was at 3000 feet, or less? Why?
Activity 3.17.
As pictured in the applet at http://gvsu.edu/s/9q, a skateboarder who is 6 feet tall rides under a 15 foot tall lamppost at a constant rate of 3 feet per second. We are interested in understanding how fast his shadow is changing at various points in time.

(a) Draw an appropriate right triangle that represents a snapshot in time of the skateboarder, lamppost, and his shadow. Let $x$ denote the horizontal distance from the base of the lamppost to the skateboarder and $s$ represent the length of his shadow. Label these quantities, as well as the skateboarder’s height and the lamppost’s height on the diagram.

(b) Observe that the skateboarder and the lamppost represent parallel line segments in the diagram, and thus similar triangles are present. Use similar triangles to establish an equation that relates $x$ and $s$.

(c) Use your work in (b) to find an equation that relates $\frac{dx}{dt}$ and $\frac{ds}{dt}$.

(d) At what rate is the length of the skateboarder’s shadow increasing at the instant the skateboarder is 8 feet from the lamppost?

(e) As the skateboarder’s distance from the lamppost increases, is his shadow’s length increasing at an increasing rate, increasing at a decreasing rate, or increasing at a constant rate?

(f) Which is moving more rapidly: the skateboarder or the tip of his shadow? Explain, and justify your answer.
Activity 3.18.

A baseball diamond is 90’ square. A batter hits a ball along the third base line runs to first base. At what rate is the distance between the ball and first base changing when the ball is halfway to third base, if at that instant the ball is traveling 100 feet/sec? At what rate is the distance between the ball and the runner changing at the same instant, if at the same instant the runner is 1/8 of the way to first base running at 30 feet/sec?
Voting Questions

3.6.1 If \( \frac{dy}{dx} = 5 \) and \( \frac{dx}{dt} = -2 \) then \( \frac{dy}{dt} = \)
(a) 5   (b) -2   (c) -10   (d) cannot be determined from the information given

3.6.2 If \( \frac{dz}{dx} = 12 \) and \( \frac{dy}{dx} = 2 \) then \( \frac{dz}{dy} = \)
(a) 24   (b) 6   (c) 1/6   (d) cannot be determined from the information given

3.6.3 If \( y = 5x^2 \) and \( \frac{dz}{dx} = 3 \), then when \( x = 4 \), \( \frac{dy}{dx} = \)
(a) 30   (b) 80   (c) 120   (d) 15x^2   (e) cannot be determined from the information given

3.6.4 The radius of a snowball changes as the snow melts. The instantaneous rate of change in radius with respect to volume is
(a) \( \frac{dV}{dr} \)   (b) \( \frac{dr}{dV} \)   (c) \( \frac{dV}{dr} + \frac{dr}{dV} \)   (d) None of the above

3.6.5 Gravel is poured into a conical pile. The rate at which gravel is added to the pile is
(a) \( \frac{dV}{dt} \)   (b) \( \frac{dr}{dt} \)   (c) \( \frac{dV}{dr} \)   (d) None of the above

3.6.6 Suppose a deli clerk can slice a stick of pepperoni so that its length \( L \) changes at a rate of 2 inches per minute and the total weight \( W \) of pepperoni that has been cut increases at a rate of 0.2 pounds per minute. The pepperoni weighs:
(a) 0.4 pounds per inch   (b) 0.1 pounds per inch   (c) 10 pounds per inch   (d) 2.2 pounds per inch   (e) None of the above

3.6.7 The area of a circle, \( A = \pi r^2 \), changes as its radius changes. If the radius changes with respect to time, the change in area with respect to time is
(a) \( \frac{dA}{dt} = 2\pi r \)   (b) \( \frac{dA}{dt} = 2\pi r + \frac{dr}{dt} \)   (c) \( \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \)   (d) Not enough information

3.6.8 As gravel is being poured into a conical pile, its volume \( V \) changes with time. As a result, the height \( h \) and radius \( r \) also change with time. Knowing that at any moment \( V = \frac{1}{3}\pi r^2 h \), the relationship between the changes in the volume, radius and height, with respect to time, is
(a) \( \frac{dV}{dt} = \frac{1}{3}\pi \left( 2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right) \)
(b) \( \frac{dV}{dt} = \frac{1}{3}\pi \left( 2r \frac{dr}{dt} \cdot \frac{dh}{dt} \right) \)
(c) \( \frac{dV}{dt} = \frac{1}{3}\pi \left( 2rh + r^2 \frac{dh}{dt} \right) \)
(d) \( \frac{dV}{dt} = \frac{1}{3}\pi \left( r^2 (1) + 2r \frac{dr}{dt} h \right) \)
3.6.9 A boat is drawn close to a dock by pulling in a rope as shown. How is the rate at which the rope is pulled in related to the rate at which the boat approaches the dock?

(a) One is a constant multiple of the other.
(b) They are equal.
(c) It depends on how close the boat is to the dock.

3.6.10 A boat is drawn close to a dock by pulling in the rope at a constant rate. The closer the boat gets to the dock, the faster it is moving. (see figure from the previous problem)

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.6.11 A streetlight is mounted at the top of a pole. A man walks away from the pole. How are the rate at which he walks away from the pole and the rate at which his shadow grows related?

(a) One is a constant multiple of the other.
(b) They are equal.
(c) It depends also on how close the man is to the pole.

3.6.12 A spotlight installed in the ground shines on a wall. A woman stands between the light and the wall casting a shadow on the wall. How are the rate at which she walks away from the light and rate at which her shadow grows related?

(a) One is a constant multiple of the other.
(b) They are equal.
(c) It depends also on how close the woman is to the pole.
3.7 Global Optimization

Preview Activity 3.7. Consider the function \( f(x) = x^4 + x^3 - 7x^2 - x + 6 \).

(a) Find \( f'(x) \) and graph both \( f(x) \) and \( f'(x) \) on the same coordinate axes.

(b) Find the critical points of \( f(x) \) and use calculus to determine if these critical points are maximums or minimums on the function \( f(x) \).

(c) Now let’s create \( g(x) = f(x) + (2x + 3) \); that is, we create \( g(x) \) by adding a linear function with a slope of 2 to the function \( f(x) \).

   (i) What is the slope of the tangent line to \( g(x) \) at the critical points from \( f(x) \)? Show all of your necessary work.

   (ii) Write a few clear sentences stating why it wasn’t actually necessary to do any calculation to answer the preview question.
Activity 3.19.
Consider the equation \(6x^5 - 15x^4 + 20x^3 - 30x^2 + 30x = 30\).

(a) Write the equation as \(f(x) = 0\), where \(f\) is a polynomial with leading term \(6x^5\).

(b) Find \(f'(x)\) and verify that \(f'(x) = 30(x^2 + 1)(x - 1)^2\).

(c) What is the maximum number of roots that \(f\) can have? What is the maximum number of solutions that the equation can have? Justify your answer using Rolle’s Theorem.
**Activity 3.20.**

In this activity we will prove the Mean Value Theorem based on Rolle’s Theorem. It might be worthwhile at this point to read again the comment relating both theorems which appeared just before the Mean Value Theorem was stated.

(a) Recall that the equation of a line with slope \( m \) going through the point \((x_1, y_1)\) can be written using the point-slope form

\[
y = m(x - x_1) + y_1.
\]

Find the slope of the secant line joining the points \((a, f(a))\) and \((b, f(b))\) and find the equation \(y = s(x)\) of this secant line.

(b) The distance, to a sign, between a point \((x, f(x))\) on the graph of function \(f\) and a point \((x, s(x))\) on the secant line can be obtained by taking the difference between the \(y\)-coordinates of the points as shown below. Create a function \(h\) which computes the distance, to a sign, between the two points.

(c) Explain why \(h\) is continuous on \([a, b]\).

(d) Explain why \(h\) is differentiable on \((a, b)\).

(e) Evaluate \(h(a)\) and \(h(b)\).

(f) Apply Rolle’s Theorem to obtain the Mean Value Theorem.

(g) The distance between Quebec City and Montreal is 233 km. A train going from Quebec City to Montreal has the following position function

\[
s(t) = 233e^{-\frac{(t-3)^2}{(1.5)^2}}.
\]
(i) Find the average speed of the train during the first three hours.
(ii) Use a graphing device or Figure 3.8 to find approximately when the speed of the train is the same as the average speed during the first three hours.
3.7. GLOBAL OPTIMIZATION

Activity 3.21.

As mentioned, the Mean Value Theorem is mostly used to prove other results. In this activity we shall demonstrate a result that will be of use in integral calculus.

(a) Let \( x_i \) and \( x_{i+1} \) be two values on the \( x \)-axis, where \( x_i < x_{i+1} \). Suppose that \( F \) is a function that satisfies the hypotheses of the Mean Value Theorem on the interval \([x_i, x_{i+1}]\) and that the derivative of \( F \) is \( f \), that is \( F' = f \). Show that the conclusion of the Mean Value Theorem is that there exists \( x^*_i \in (x_i, x_{i+1}) \) such that

\[
F(x_{i+1}) - F(x_i) = f(x^*_i)(x_{i+1} - x_i). 
\]  
(3.1)

(b) Consider the interval \([1, 9]\).

(i) Divide the interval in 4 equal parts, that is divide the interval in 4 subintervals of equal length.

(ii) Let the 4 subintervals be \([x_0, x_1]\), \([x_1, x_2]\), \([x_2, x_3]\), and \([x_3, x_4]\). Find the values of \( x_0, x_1, x_2, x_3, \) and \( x_4 \).

(iii) We can write the difference \( F(9) - F(1) = F(x_4) - F(x_0) \) as

\[
F(9) - F(1) = F(x_4) - F(x_0), \\
= F(x_4) - F(x_3) + F(x_3) - F(x_2) + F(x_2) - F(x_1) + F(x_1) - F(x_0), \\
= [F(x_4) - F(x_3)] + [F(x_3) - F(x_2)] + [F(x_2) - F(x_1)] + [F(x_1) - F(x_0)], \\
= [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + [F(x_3) - F(x_2)] + [F(x_4) - F(x_3)].
\]

Since the subintervals have equal length, the difference between two consecutive values \( x_i \) is constant. Let the difference be \( \Delta x \), that is \( \Delta x = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3 \). Apply the Mean Value Theorem (see Equation 3.1) to each bracket in the last expression to justify the following equation

\[
F(9) - F(1) = f(x^*_1)\Delta x + f(x^*_2)\Delta x + f(x^*_3)\Delta x + f(x^*_4)\Delta x,
\]

where \( x^*_i \in (x_{i-1}, x_i) \).

(c) (Hard) Consider the interval \([a, b]\) and divide the interval in \( n \) subintervals of equal length \( \Delta x = (b - a)/n \) and let \( x_0, x_1, \ldots, x_{n-1}, x_n \) be the endpoints of the subintervals. Show that

\[
F(b) - F(a) = f(x^*_1)\Delta x + f(x^*_2)\Delta x + \ldots + f(x^*_n)\Delta x,
\]

where \( x^*_i \in (x_{i-1}, x_i) \). As a concluding remark, the limit

\[
\lim_{n \to \infty} f(x^*_1)\Delta x + f(x^*_2)\Delta x + \ldots + f(x^*_n)\Delta x
\]

is called a *definite integral* and it represents the net area between the function \( f \) and the \( x \)-axis for \( a \leq x \leq b \). What this exercise has shown is that this difficult limit problem can be solved by simply evaluating the net change \( F(b) - F(a) \). This result is called the *Fundamental Theorem of Calculus* and is one of the most important theorems in integral calculus/Calculus II (which you will be doing next semester), but also one of the most important theorems in mathematics in general.
Voting Questions