Active Calculus & Mathematical Modeling
Activities and Voting Questions
Carroll College

Carroll College Mathematics Department

Last Update: August 15, 2017
To The Student

This packet is NOT your textbook. What you will find here are

- **Preview Activities**
  These questions are designed to be done before class. They are meant to give you an accessible preview of the material covered in a section of the text.

- **Activities**
  These questions are meant to teach and re-enforce material presented in a section of the book. Many of these questions will be discussed during class time.

- **Voting Questions**
  These are multiple choice and true/false questions that will be used with the classroom clicker technology. Think of these as concept questions that address new definitions, new techniques, common misconceptions, and good review.

You should bring this booklet to class every day, but again remember that

**THIS BOOKLET IS NOT THE TEXTBOOK FOR THE CLASS.**

Instead, this booklet contains supplemental material.

To access the full textbook go to the Moodle page for your class or www.carroll.edu/mathtexts/. I’ll wait while you go to that website and have a look

... OK. Now that you’ve looked at the text you’ll know where to go to read more about the material being presented in class. If you want to print out the text then the costs are on you.
## Contents

0 Preliminaries ........................................ 7
  0.1 Functions, Slope, and Lines ........................... 7
  0.2 Exponential Functions ................................ 21
  0.3 Transformations of Functions ......................... 26
  0.4 Logarithmic Functions ................................ 40
  0.5 Trigonometric Functions ............................... 53
  0.6 Powers, Polynomials, and Rational Functions ....... 64

1 Understanding the Derivative .......................... 77
  1.1 How do we measure velocity? ......................... 77
  1.2 The notion of limit .................................. 85
  1.3 The derivative of a function at a point ............... 93
  1.4 The derivative function ............................... 99
  1.5 Interpreting the derivative and its units ............. 107
  1.6 The second derivative ................................ 113
  1.7 Limits, Continuity, and Differentiability .......... 123
  1.8 The Tangent Line Approximation ...................... 131

2 Computing Derivatives ................................ 139
  2.1 Elementary derivative rules ......................... 139
  2.2 The sine and cosine functions ....................... 152
  2.3 The product and quotient rules ....................... 158
  2.4 Derivatives of other trigonometric functions ...... 167
  2.5 The chain rule ...................................... 172
CONTENTS

2.6 Derivatives of Inverse Functions ......................................................... 180
2.7 Derivatives of Functions Given Implicitly ............................................... 186
2.8 Using Derivatives to Evaluate Limits .................................................... 191
2.9 More on Limits ...................................................................................... 199

3 Using Derivatives ................................................................. 205
3.1 Using derivatives to identify extreme values of a function ....................... 205
3.2 Using derivatives to describe families of functions .................................. 212
3.3 Global Optimization ........................................................................... 218
3.4 Introduction to Sensitivity Analysis ....................................................... 224
3.5 Applied Optimization .......................................................................... 228
3.6 Related Rates ....................................................................................... 234
3.7 Global Optimization ........................................................................... 241

4 The Definite Integral ................................................................. 247
4.1 Determining distance traveled from velocity ........................................... 247
4.2 Riemann Sums ..................................................................................... 255
4.3 The Definite Integral ........................................................................... 261
4.4 The Fundamental Theorem of Calculus & Basic Antiderivatives .............. 268

5 Finding Antiderivatives and Evaluating Integrals ................................ 283
5.1 Constructing Accurate Graphs of Antiderivatives .................................... 283
5.2 The Second Fundamental Theorem of Calculus .................................... 297
5.3 Integration by Substitution ..................................................................... 305
5.4 Integration by Parts .............................................................................. 315
5.5 Other Options for Finding Algebraic Antiderivatives ............................... 322
5.6 Numerical Integration ........................................................................... 326

6 Using Definite Integrals ................................................................. 331
6.1 Using Definite Integrals to Find Area and Length .................................... 331
6.2 Using Definite Integrals to Find Volume ................................................ 336
6.3 Density, Mass, and Center of Mass ....................................................... 345
6.4 Physics Applications: Work, Force, and Pressure ................................... 350
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.5</td>
<td>Improper Integrals</td>
<td>356</td>
</tr>
<tr>
<td>7</td>
<td>Differential Equations</td>
<td>363</td>
</tr>
<tr>
<td>7.1</td>
<td>An Introduction to Differential Equations</td>
<td>363</td>
</tr>
<tr>
<td>7.2</td>
<td>Qualitative behavior of solutions to differential equations</td>
<td>371</td>
</tr>
<tr>
<td>7.3</td>
<td>Euler’s method</td>
<td>382</td>
</tr>
<tr>
<td>7.4</td>
<td>Separable differential equations</td>
<td>394</td>
</tr>
<tr>
<td>7.5</td>
<td>Modeling with differential equations</td>
<td>401</td>
</tr>
<tr>
<td>7.6</td>
<td>Population Growth and the Logistic Equation</td>
<td>405</td>
</tr>
<tr>
<td>7.7</td>
<td>Linear First Order Differential Equation (Constant Coefficients)</td>
<td>418</td>
</tr>
<tr>
<td>7.8</td>
<td>Linear Second Order Differential Equations (Mass Spring Systems)</td>
<td>427</td>
</tr>
<tr>
<td>7.9</td>
<td>Forced Oscillations</td>
<td>437</td>
</tr>
<tr>
<td>8</td>
<td>Sequences and Series</td>
<td>439</td>
</tr>
<tr>
<td>8.1</td>
<td>Sequences</td>
<td>439</td>
</tr>
<tr>
<td>8.2</td>
<td>Geometric Series</td>
<td>445</td>
</tr>
<tr>
<td>8.3</td>
<td>Series of Real Numbers</td>
<td>452</td>
</tr>
<tr>
<td>8.4</td>
<td>Alternating Series</td>
<td>467</td>
</tr>
<tr>
<td>8.5</td>
<td>Taylor Polynomials and Taylor Series</td>
<td>475</td>
</tr>
<tr>
<td>8.6</td>
<td>Power Series</td>
<td>483</td>
</tr>
<tr>
<td>8.7</td>
<td>Functional DNA</td>
<td>490</td>
</tr>
</tbody>
</table>
Chapter 0

Preliminaries

0.1 Functions, Slope, and Lines

Preview Activity 0.1. This is the first Preview Activity in this text. Your job for this activity is to get to know the textbook.

(a) Where can you find the full textbook?

(b) What chapters of this text are you going to cover this semester. Have a look at your syllabus!

(c) What are the differences between Preview Activities, Activities, Examples, Exercises, Voting Questions, and WeBWork? Which ones should you do before class, which ones will you likely do during class, and which ones should you be doing after class?

(d) What materials in this text would you use to prepare for an exam and where do you find them?

(e) What should you bring to class every day?
Activity 0.1.

The graph of a function \( f(x) \) is shown in the plot below.

(a) What is the domain of \( f(x) \)?
(b) Approximate the range of \( f(x) \).
(c) What are \( f(0), f(1), f(3), f(4), \) and \( f(5) \)?
Activity 0.2.

Find the equation of the line with the given information.

(a) The line goes through the points \((-2, 5)\) and \((10, -1)\).

(b) The slope of the line is \(\frac{3}{5}\) and it goes through the point \((2, 3)\).

(c) The \(y\)-intercept of the line is \((0, -1)\) and the slope is \(-\frac{2}{3}\).
Activity 0.3.

An apartment manager keeps careful record of the rent that he charges as well as the number of occupied apartments in his complex. The data that he has is shown in the table below.

<table>
<thead>
<tr>
<th>Monthly Rent</th>
<th>$650</th>
<th>$700</th>
<th>$750</th>
<th>$800</th>
<th>$850</th>
<th>$900</th>
</tr>
</thead>
<tbody>
<tr>
<td>Occupied Apartments</td>
<td>203</td>
<td>196</td>
<td>189</td>
<td>182</td>
<td>175</td>
<td>168</td>
</tr>
</tbody>
</table>

(a) Just by doing simple arithmetic justify that the function relating the number of occupied apartments and the rent is linear.

(b) Find the linear function relating the number of occupied apartments to the rent.

(c) If the rent were to be increased to $1000, how many occupied apartments would the apartment manager expect to have?

(d) At a $1000 monthly rent what net revenue should the apartment manager expect?
Activity 0.4.
Write the equation of the line with the given information.

(a) Write the equation of a line parallel to the line \( y = \frac{1}{2}x + 3 \) passing through the point \((3, 4)\).

(b) Write the equation of a line perpendicular to the line \( y = \frac{1}{2}x + 3 \) passing through the point \((3, 4)\).

(c) Write the equation of a line with \( y \)-intercept \((0, -3)\) that is perpendicular to the line \( y = -3x - 1 \).
Voting Questions

0.1.1 In the given equation, is $y$ a function of $x$?

$$y = x + 2$$

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.2 In the given equation, is $y$ a function of $x$?

$$x + y = 5$$

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.3 In the given equation, is $y$ a function of $x$?

$$x^3 + y = 5$$

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.4 In the given equation, is $y$ a function of $x$?

$$x^2 + y^2 = 5$$

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.5 The set of points \((x, y)\) which satisfy the equation \((x - 1)^2 + (y + 3)^2 = 5^2\) can be represented via a mathematical function relating the \(x\) and \(y\) variables.

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.1.6 Does the table represent a function, \(y = f(x)\)?

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.7 Does the table represent a function, \(y = f(x)\)?

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.8 Does this sentence describe a function? Wanda is two years older than I am.

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident
0.1.9 The rule which assigns to each college student (at this exact point in time) a number equal to the number of college credits completed by that student is a function.

(a) True, and I am very confident.  
(b) True, but I am not very confident.  
(c) False, but I am not very confident.  
(d) False, and I am very confident.

0.1.10 The rule which assigns to each car (at this exact point in time) the names of every person that has driven that car is a function.

(a) True, and I am very confident.  
(b) True, but I am not very confident.  
(c) False, but I am not very confident.  
(d) False, and I am very confident.

0.1.11 Could this table represent a linear function?

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident  
(b) Yes, but I am not very confident  
(c) No, but I am not very confident  
(d) No, and I am very confident

0.1.12 Could this table represent a linear function?

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-12</td>
<td>-9</td>
<td>-6</td>
<td>-3</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident  
(b) Yes, but I am not very confident  
(c) No, but I am not very confident  
(d) No, and I am very confident
0.1.13 Could this table represent a linear function?

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.14 Could this table represent a linear function?

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>10</td>
<td>9</td>
<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Yes, and I am very confident
(b) Yes, but I am not very confident
(c) No, but I am not very confident
(d) No, and I am very confident

0.1.15 True or False? All linear functions are examples of direct proportionality.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

0.1.16 Find the domain of the function $f(x) = \frac{1}{x-2}$.

(a) $x = 2$
(b) $x \neq 2$
(c) $x < 2$
(d) all real numbers

0.1.17 Find the domain of the function $g(t) = \frac{2+t}{\sqrt{t-7}}$.

(a) $t > 7$
(b) $t \geq 7$
0.1.18 Which of the following functions has its domain identical with its range?

(a) \( f(x) = x^2 \)
(b) \( g(x) = \sqrt{x} \)
(c) \( h(x) = x^4 \)
(d) \( i(x) = |x| \)

0.1.19 The slope of the line connecting the points (1,4) and (3,8) is

(a) \(-\frac{1}{2}\)
(b) \(-2\)
(c) \(\frac{1}{2}\)
(d) 2

0.1.20 Which one of these lines has a different slope than the others?

(a) \( y = 3x + 2 \)
(b) \( 3y = 9x + 4 \)
(c) \( 3y = 3x + 6 \)
(d) \( 2y = 6x + 4 \)

0.1.21 The graph below represents which function?

(a) \( y = 6x + 6 \)
(b) \( y = -3x + 6 \)
(c) \( y = -3x + 2 \)
(d) \( y = -x + 6 \)
(e) \( y = 6x - 2 \)
(f) \( y = x - 2 \)
0.1.22 Which of the following functions is not increasing?

(a) The elevation of a river as a function of distance from its mouth
(b) The length of a single strand of hair as a function of time
(c) The height of a person from age 0 to age 80
(d) The height of a redwood tree as a function of time

0.1.23 Which of these graphs does not represent $y$ as a function of $x$?

Plot (a)

Plot (b)

Plot (c)

Plot (d)

0.1.24 Calculate the average rate of change of the function $f(x) = x^2$ between $x = 1$ and $x = 3$.

(a) 8
(b) 4
0.1.25 The EPA reports the total amount of Municipal Solid Waste (MSW), otherwise known as garbage, produced in the U.S. for the years 2005 through 2009:

<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Millions of tons</td>
<td>252.4</td>
<td>251.3</td>
<td>255</td>
<td>249.6</td>
<td>243</td>
</tr>
</tbody>
</table>

(source: http://www.epa.gov/osw/nonhaz/municipal/)

What are the appropriate units for the average rate of change in the amount of garbage produced between any two given years?

(a) millions of tons
(b) tons
(c) millions of tons per year
(d) tons per year

0.1.26 The EPA reports the total amount of Municipal Solid Waste (MSW), otherwise known as garbage, produced in the U.S. for the years 2005 through 2009:

<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Millions of tons</td>
<td>252.4</td>
<td>251.3</td>
<td>255</td>
<td>249.6</td>
<td>243</td>
</tr>
</tbody>
</table>

(source: http://www.epa.gov/osw/nonhaz/municipal/)

What is the average rate of change in the amount of MSW produced from 2005 to 2007?

(a) 2.6 million tons per year
(b) 2.6 million tons
(c) 1.3 million tons
(d) 1.3 million tons per year

0.1.27 The EPA reports the total amount of Municipal Solid Waste (MSW), otherwise known as garbage, produced in the U.S. for the years 2005 through 2009:
What is the average rate of change in the amount of MSW produced from 2007 to 2009?

(a) $-6$ million tons per year
(b) $6$ million tons per year
(c) $-12$ million tons per year
(d) $12$ million tons per year

0.1.28 Find the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the function $f(x) = 2x^2 - x + 3$. Simplify your answer.

(a) $\frac{2h^2 - h + 3}{h}$
(b) $2h - 1$
(c) $\frac{4xh + 2h^2 - 2x + h + 6}{h}$
(d) $4x + 2h - 1$

0.1.29 When the temperature is $0^\circ C$ it is $32^\circ F$ and when it is $100^\circ C$ it is $212^\circ F$. Use these facts to write a linear function to convert any temperature from Celsius to Fahrenheit.

(a) $C(F) = \frac{5}{9}F - \frac{160}{9}$
(b) $F(C) = C + 32$
(c) $F(C) = \frac{5}{9}C - \frac{160}{9}$
(d) $F(C) = \frac{9}{5}C + 32$

0.1.30 Let $f(x) = 1 + 4x^2$. True or False: $f\left(\frac{1}{2}\right) = \frac{f(1)}{f(2)}$.

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.1.31 Let $f(x) = 1 + 4x^2$. True or False: $f(a + b) = f(a) + f(b)$.
0.1. FUNCTIONS, SLOPE, AND LINES

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.1.32 Let \( f(x) = \frac{1}{x+2} \). Find a value of \( x \) so that \( f(x) = 6 \)

(a) \(-\frac{11}{6}\)
(b) \(\frac{13}{6}\)
(c) \(\frac{1}{8}\)
(d) none of the above

0.1.33 True or False: \( \sqrt{x^2} = x \).

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.
0.2 Exponential Functions

Preview Activity 0.2. Suppose that the populations of two towns are both growing over time. The town of Exponentia is growing at a rate of 2% per year, and the town of Lineola is growing at a rate of 100 people per year. In 2014, both of the towns have 2,000 people.

(a) Complete the table for the population of each of these towns over the next several years.

<table>
<thead>
<tr>
<th>Year</th>
<th>Exponentia</th>
<th>Lineola</th>
</tr>
</thead>
<tbody>
<tr>
<td>2014</td>
<td>2000</td>
<td>2000</td>
</tr>
<tr>
<td>2015</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2016</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2017</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2018</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2019</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2020</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2021</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2022</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Write a linear function for the population of Lineola. Interpret the slope in the context of this problem.

(c) The ratio of successive populations for Exponentia should be equal. For example, dividing the population in 2015 by that of 2014 should give the same ratio as when the population from 2016 is divided by the population of 2015. Find this ratio. How is this ratio related to the 2% growth rate?

(d) Based on your data from part (a) and your ratio in part (c), write a function for the population of Exponentia.

(e) When will the population of Exponentia exceed that of Lineola?
Activity 0.5.

Consider the exponential functions plotted in Figure 1

(a) Which of the functions have common ratio $r > 1$?

(b) Which of the functions have common ratio $0 < r < 1$?

(c) Rank each of the functions in order from largest to smallest $r$ value.

Figure 1: Exponential growth and decay functions
Activity 0.6.

A sample of \(Ni^{56}\) has a half-life of 6.4 days. Assume that there are 30 grams present initially.

(a) Write a function describing the number of grams of \(Ni^{56}\) present as a function of time. Check your function based on the fact that in 6.4 days there should be 50% remaining.

(b) What percent of the substance is present after 1 day?

(c) What percent of the substance is present after 10 days?
Activity 0.7.

Uncontrolled geometric growth of the bacterium Escherichia coli (E. Coli) is the theme of the following quote taken from the best-selling author Michael Crichton’s science fiction thriller, The Andromeda Strain:

“The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth.”

(a) Write an equation for the number of E. coli cells present if a single cell of E. coli divides every 20 minutes.

(b) How many E. coli would there be at the end of 24 hours?

(c) The mass of an E. coli bacterium is $1.7 \times 10^{-12}$ grams, while the mass of the Earth is $6.0 \times 10^{27}$ grams. Is Michael Crichton’s claim accurate? Approximate the number of hours we should have allowed for this statement to be correct?
0.2. EXPONENTIAL FUNCTIONS

Voting Questions

0.2.1 The following table shows the net sales at Amazon.com from 2003 to 2010 (source: Amazon.com quarterly reports):

<table>
<thead>
<tr>
<th>Year</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bills of dollars</td>
<td>$5.26</td>
<td>$6.92</td>
<td>$8.49</td>
<td>$10.72</td>
<td>$14.84</td>
<td>$19.15</td>
<td>$24.51</td>
<td>$34.21</td>
</tr>
</tbody>
</table>

If the net sales are modeled using an exponential function $S(t) = a \cdot b^t$, where $S$ is the net sales in billions of dollars, and $t$ is the number of years after 2003, which of the following is an appropriate value for $a$?

(a) 34.21
(b) 1
(c) 1.31
(d) 5.26

0.2.2 Which is better at the end of one year: An account that pays 8% annual interest compounded quarterly or an account that pays 7.95% interest compounded continuously?

(a) 8% quarterly
(b) 7.95% continuously
(c) They are the same.
(d) There is no way to tell.

0.2.3 Caffeine leaves the body at a continuous rate of 17% per hour. How much caffeine is left in the body 8 hours after drinking a cup of coffee containing 100 mg of caffeine?

(a) 389.62 mg
(b) 22.52 mg
(c) 25.67 mg
(d) There is no way to tell.

0.2.4 Caffeine leaves the body at a continuous rate of 17% per hour. What is the hourly growth factor?

(a) .156
(b) .17
(c) .844
(d) There is no way to tell.
0.3 Transformations of Functions

Preview Activity 0.3. The goal of this activity is to explore and experiment with the function

\[ F(x) = Af(B(x - C)) + D. \]

The values of \( A, B, C, \) and \( D \) are constants and the function \( f(x) \) will be henceforth called the parent function. To facilitate this exploration, use the applet located at http://www.geogebratube.org/student/m93018.

(a) Let’s start with a simple parent function: \( f(x) = x^2 \).

(1) Fix \( B = 1, C = 0, \) and \( D = 0 \). Write a sentence or two describing the action of \( A \) on the function \( F(x) \).

(2) Fix \( A = 1, B = 1, \) and \( D = 0 \). Write a sentence of two describing the action of \( C \) on the function \( F(x) \).

(3) Fix \( A = 1, B = 1, \) and \( C = 0 \). Write a sentence of two describing the action of \( D \) on the function \( F(x) \).

(4) Fix \( A = 1, C = 0, \) and \( D = 0 \). Write a sentence of two describing the action of \( B \) on the function \( F(x) \).

(b) In part (a) you have made conjectures about what \( A, B, C, \) and \( D \) do to a parent function graphically. Test your conjectures with the functions \( f(x) = |x| \) (typed abs(x)), \( f(x) = x^3 \), \( f(x) = \sin(x) \), \( f(x) = e^x \) (typed exp(x)), and any other function you find interesting.
Activity 0.8.
Consider the function $f(x)$ displayed in Figure 2.

(a) Plot $g(x) = -f(x)$ and $h(x) = f(x) - 1$.

(b) Define the function $k(x) = -f(x) - 1$. Does it matter which order you complete the transformations from part (a) to result in $k(x)$? Plot the functions resulting from doing the two transformation in part (a) in opposite orders. Which of these functions is $k(x)$?

Figure 2: Function transformation for Activity 0.8
Activity 0.9.
(a) Let \( f(x) = x^2 \) and \( g(x) = x + 8 \). Find the following:
\[
\begin{align*}
f(g(3)) &= , \\
g(f(3)) &= , \\
f(g(x)) &= , \\
g(f(x)) &= , \\
f(x)g(x) &= .
\end{align*}
\]
(b) Now let \( f(x) \) and \( g(x) \) be defined as in the table below. Use the data in the table to find the following compositions.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
f(-3) &= , \\
g(3) &= ,
\end{align*}
\]
\[
\begin{align*}
f(g(-3)) &= , \\
f(g(f(-3))) &= .
\end{align*}
\]
(c) Now let \( f(x) \) and \( g(x) \) be defined as in the plots below. Use the plots to find the following compositions.

\[
\begin{align*}
f(1) &= , \\
g(2) &= ,
\end{align*}
\]
\[
\begin{align*}
g(f(1)) &= , \\
f(g(1)) &= , \\
g(f(f(0))) &= .
\end{align*}
\]
Activity 0.10.

(a) Based on symmetry alone, is \( f(x) = x^2 \) an even or an odd function?

(b) Based on symmetry alone, is \( g(x) = x^3 \) an even or an odd function?

(c) Find \( f(-x) \) and \( g(-x) \) and make conjectures to complete these sentences:
   - If a function \( f(x) \) is even then \( f(-x) = \) ___________.
   - If a function \( f(x) \) is odd then \( f(-x) = \) ___________.

Explain why the composition \( f(-x) \) is a good test for symmetry of a function.

(d) Classify each of the following functions as even, odd, or neither.

\[
\begin{align*}
h(x) &= \frac{1}{x}, & j(x) &= e^x, & k(x) &= x^2 - x^4, & n(x) &= x^3 + x^2.
\end{align*}
\]

(e) Each figure below shows only half of the function. Draw the left half so \( f(x) \) is even. Draw the left half so \( g(x) \) is odd. Draw the left half so \( h(x) \) is neither even nor odd.
Activity 0.11.

(a) Find the inverse of each of the following functions by interchanging the $x$ and $y$ and solving for $y$. Be sure to state the domain for each of your answers.

\[ y = \sqrt{x - 1}, \quad y = -\frac{1}{3}x + 1, \quad y = \frac{x + 4}{2x - 5} \]

(b) Verify that the functions $f(x) = 3x - 2$ and $g(x) = \frac{2}{3} + \frac{2}{3}$ are inverses of each other by computing $f(g(x))$ and $g(f(x))$. 

\[ \triangleq \]
0.3. TRANSFORMATIONS OF FUNCTIONS

Voting Questions

0.3.1 The functions $f$ and $g$ have values given in the table below. What is the value of $f(g(0))$?

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

(a) -2  
(b) -1  
(c) 0  
(d) 1  
(e) 2

0.3.2 The functions $f$ and $g$ have values given in the table below. If $f(g(x)) = 1$, then what is $x$?

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

(a) -2  
(b) -1  
(c) 0  
(d) 1  
(e) 2

0.3.3 The graphs of $f$ and $g$ are shown in the figure below. Estimate the value of $g(f(3))$.

(a) -1  
(b) 0  
(c) 1  
(d) 2  
(e) 3  
(f) 5
0.3.4 The graphs of \( f \) and \( g \) are shown in the figure below. Estimate the value of \( f(g(2)) \).

\[ (a) \ -1 \quad (b) \ 0 \quad (c) \ 1 \quad (d) \ 2 \quad (e) \ 3 \quad (f) \ 5 \]

0.3.5 If \( P = f(t) = 3 + 4t \), find \( f^{-1}(P) \).

\[ (a) \ f^{-1}(P) = 3 + 4P \]
\[ (b) \ f^{-1}(P) = \frac{P-3}{4} \]
\[ (c) \ f^{-1}(P) = \frac{P-4}{3} \]
\[ (d) \ f^{-1}(P) = 4(P + 3) \]
\[ (e) \ f^{-1}(P) = \frac{P+3}{4} \]

0.3.6 Which of these functions has an inverse?

Plot (a)  

Plot (b)
0.3. TRANSFORMATIONS OF FUNCTIONS

The following is a graph of \( f(x) \). Which graph below is the inverse?

(a) (a) only
(b) (b) only
(c) (c) only
(d) (d) only
(e) (a) and (b)
(f) (b) and (c)
0.3.8 Given that \( f(x) = \sqrt{\frac{x^3 - 72}{800}} \), find \( f \circ f^{-1}(437) \).

(a) 104.31673
(b) 1671.2
(c) 437
(d) 10.08

0.3.9 If \( f(x) = \frac{x}{x^2 + 1} \), what is \( f^{-1} \circ f(-2) \)?

(a) \( \frac{2}{5} \)
(b) \( \frac{2}{3} \)
0.3.10 If (4, -2) is a point on the graph of \( y = f(x) \), which of the following points is on the graph of \( y = f^{-1}(x) \)?

(a) (-2, 4)
(b) (-4, 2)
(c) \( \left( \frac{1}{4}, -\frac{1}{2} \right) \)
(d) \( \left( -\frac{1}{4}, \frac{1}{2} \right) \)

0.3.11 Find the inverse of \( f(x) = \frac{1}{x} \).

(a) \( f^{-1}(x) = \frac{x}{1} \)
(b) \( f^{-1}(x) = x \)
(c) \( f^{-1}(x) = \frac{1}{x} \)
(d) \( f^{-1}(x) = xy \)

0.3.12 A function is given in Figure 1.10 below. Which one of the other graphs could be a graph of \( f(x + h) \)?
0.3.13 How is the graph of \( y = 2^{x-1} + 3 \) obtained from the graph of \( y = 2^x \)?

(a) Move 1 down and 3 right
(b) Move 1 left and 3 up
(c) Move 1 up and 3 right
(d) Move 1 right and 3 up

0.3.14 The function \( f(x) \) goes through the point \( A \) with coordinates \((2,3)\). \( g(x) = 2f\left(\frac{1}{3}x - 2\right) + 4 \). What are the coordinates of point \( A \) in the function \( g(x) \)?

(a) \((4, 10)\)
(b) \((4, -\frac{5}{2})\)
0.3. TRANSFORMATIONS OF FUNCTIONS

0.3.15 The point (4, 1) is on the graph of a function \( f \). Find the corresponding point on the graph of \( y = f(x - 2) \).

(a) (6, 1)
(b) (2, 1)
(c) (4, 3)
(d) (4, −1)

0.3.16 The point (6, 1) is on the graph of a function \( f \). Find the corresponding point on the graph of \( y = f(2x) \).

(a) (12, 1)
(b) (3, 1)
(c) (6, 2)
(d) (6, \( \frac{1}{2} \))

0.3.17 Given the graph of a function \( f(x) \), what sequence of activities best describes the process you might go through to graph \( g(x) = 5f(-x) \)?

(a) Expand the graph by a factor of 5, then reflect it across the \( y \)-axis.
(b) Expand the graph by a factor of 5, then reflect it across the \( x \)-axis.
(c) Reflect the graph across the \( y \)-axis, then expand it by a factor of 5.
(d) Reflect the graph across the \( x \)-axis, then expand it by a factor of 5.
(e) More than 1 of the above.
(f) None of the above.

0.3.18 Given the graph of a function \( f(x) \), what sequence of activities best describes the process you might go through to graph \( g(x) = -f(x) + 2 \)?

(a) Move the graph up 2 units, then reflect it across the \( x \)-axis.
(b) Move the graph up 2 units, then reflect it across the \( y \)-axis.
(c) Reflect the graph across the $y$-axis, then move it up by 2 units.
(d) Reflect the graph across the $x$-axis, then move it up 2 units.
(e) More than 1 of the above.
(f) None of the above.

0.3.19 Take the function $f(x)$ and “Shift the function right $h$ units. Reflect the result across the $y$-axis, then reflect the result across the $x$-axis. Finally shift the result up $k$ units.” The end result is:
(a) $f(x + h) + k$
(b) $f(x - h) + k$
(c) $-f(-x - h) + k$
(d) $-f(-x + h) + k$

0.3.20 Given $f(x) = x + 1$ and $g(x) = 3x^2 - 2x$, what is the composition $g(f(x))$.
(a) $3x^2 - 2x + 1$
(b) $(3x^2 - 2x)(x + 1)$
(c) $3x^2 + 4x + 1$
(d) $3(x + 1)^2 - 2x$

0.3.21 Write $h(x) = e^{3x/2}$ as a composition of functions: $f(g(x))$. $f(x) =$ ____________, $g(x) =$ ____________.
(a) $e^x, 3x/2$
(b) $3x/2, e^x$
(c) $x, e^{3x/2}$
(d) $x/2, 3e^x$

0.3.22 If $f(x) = x^2 + 6$ and $g(x) = x - 3$, what is $f \circ g(x)$?
(a) $x^2 + 3$
(b) $x^2 - 6x + 15$
(c) $x^2 - 3$
(d) $x^3 - 3x^2 + 6x - 18$
0.3.23 Which of the following functions is invertible?

(a) \( f(x) = -x^4 + 7 \)
(b) \( g(x) = e^{3x/2} \)
(c) \( h(x) = \cos(x) \)
(d) \( k(x) = |x| \)

0.3.24 Let \( f(x) = x - 2 \) and \( g(x) = 3 - x^2 \). Find \( g(f(2)) \).

(a) -3
(b) 0
(c) 3
(d) 2

0.3.25 If \( P = f(t) = 3 + 4t \), find \( f^{-1}(7) \).

(a) 31
(b) \( \frac{1}{7} \)
(c) 0
(d) 1

0.3.26 Let \( f(x) = x^2 \) and \( g(x) = x + 2 \). True or false? The domain of the function \( \frac{f}{g} \) is \( \mathbb{R} \), all real numbers.

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.3.27 Let \( f(x) = x^2 - 4 \) and \( g(x) = \sqrt{x} \). Find \( (g \circ f)(x) \) and the domain of \( g \circ f \).

(a) \( \sqrt{x^2 - 4} \); Domain: \((-\infty, -2] \cup [2, \infty)\)
(b) \( x - 4 \); Domain: \( \mathbb{R} \)
(c) \( x - 4 \); Domain: \([0, \infty)\)
(d) \( \sqrt{x^2 - 4} \); Domain: \([0, \infty)\)
(e) \( \sqrt{4x^2 - 4} \); Domain: \([0, \infty)\)
0.4 Logarithmic Functions

Preview Activity 0.4. Carbon-14 ($^{14}$C) is a radioactive isotope of carbon that occurs naturally in the Earth’s atmosphere. During photosynthesis, plants take in $^{14}$C along with other carbon isotopes, and the levels of $^{14}$C in living plants are roughly the same as atmospheric levels. Once a plant dies, it no longer takes in any additional $^{14}$C. Since $^{14}$C in the dead plant decays at a predictable rate (the half-life of $^{14}$C is approximately 5,730 years), we can measure $^{14}$C levels in dead plant matter to get an estimate on how long ago the plant died. Suppose that a plant has 0.02 milligrams of $^{14}$C when it dies.

(a) Write a function that represents the amount of $^{14}$C remaining in the plant after $t$ years.

(b) Complete the table for the amount of $^{14}$C remaining $t$ years after the death of the plant.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>100</th>
<th>2000</th>
<th>5730</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{14}$C Level</td>
<td>0.02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Suppose our plant died sometime in the past. If we find that there are 0.014 milligrams of $^{14}$C present in the plant now, estimate the age of the plant to within 50 years.
Activity 0.12.

Use the definition of a logarithm along with the properties of logarithms to answer the following.

(a) Write the exponential expression $8^{1/3} = 2$ as a logarithmic expression.
(b) Write the logarithmic expression $\log_2 \frac{1}{32} = -5$ as an exponential expression.
(c) What value of $x$ solves the equation $\log_2 x = 3$?
(d) What value of $x$ solves the equation $\log_2 4 = x$?
(e) Use the laws of logarithms to rewrite the expression $\log (x^3 y^5)$ in a form with no logarithms of products, quotients, or powers.
(f) Use the laws of logarithms to rewrite the expression $\log \left( \frac{x^{15} y^{20}}{z^4} \right)$ in a form with no logarithms of products, quotients, or powers.
(g) Rewrite the expression $\ln(8) + 5 \ln(x) + 15 \ln(x^2 + 8)$ as a single logarithm.
Activity 0.13.

Solve each of the following equations for \( t \), and verify your answers using a calculator.

(a) \( \ln t = 4 \)
(b) \( \ln(t + 3) = 4 \)
(c) \( \ln(t + 3) = \ln 4 \)
(d) \( \ln(t + 3) + \ln(t) = \ln 4 \)
(e) \( e^t = 4 \)
(f) \( e^{t+3} = 4 \)
(g) \( 2e^{t+3} = 4 \)
(h) \( 2e^{3t+2} = 3e^{t-1} \)
Activity 0.14.

Consider the following equation:

\[ 7^x = 24 \]

(a) How many solutions should we expect to find for this equation?

(b) Solve the equation using the log base 7.

(c) Solve the equation using the log base 10.

(d) Solve the equation using the natural log.

(e) Most calculators have buttons for \( \log_{10} \) and \( \ln \), but none have a button for \( \log_7 \). Use your previous answers to write a formula for \( \log_7 x \) in terms of \( \log x \) or \( \ln x \).
Activity 0.15.

(a) In the presence of sufficient resources the population of a colony of bacteria exhibits exponential growth, doubling once every three hours. What is the corresponding continuous (percentage) growth rate?

(b) A hot bowl of soup is served at a dinner party. It starts to cool according to Newton’s Law of Cooling so its temperature, $T$ (measured in degrees Fahrenheit) after $t$ minutes is given by

$$T(t) = 65 + 186e^{-0.06t}.$$  

How long will it take from the time the food is served until the temperature is 120°F?

(c) The velocity (in ft/sec) of a sky diver $t$ seconds after jumping is given by

$$v(t) = 80 \left(1 - e^{-0.2t}\right).$$

After how many seconds is the velocity 75 ft/sec?
Voting Questions

0.4.1 A logarithmic function of the form \( f(x) = \log_a x \) will always pass through the point \((1, 0)\).

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.4.2 Which is a graph of \( y = \ln x \)?

![Graphs of logarithmic functions]

Plot (a) and Plot (b) are incorrect as they do not represent the graph of \( y = \ln x \). Plot (c) and Plot (d) are correct representations of the graph of \( y = \ln x \).
0.4.3 The graph below could be that of

(a) $y = \ln x + \frac{1}{2}$
(b) $y = \ln x - \frac{1}{2}$
(c) $y = \ln (x + \frac{1}{2})$
(d) $y = \ln (x - \frac{1}{2})$

0.4.4 Which equation matches this graph?

(a) $y = b^x$ with $b > 1$
(b) $y = b^x$ with $0 < b < 1$
(c) $y = \log_b x$ with $b > 1$
(d) $y = \log_b x$ with $0 < b < 1$

0.4.5 Which equation matches this graph?

(a) $y = b^x$ with $b > 1$
(b) $y = b^x$ with $0 < b < 1$
(c) $y = \log_b x$ with $b > 1$
(d) $y = \log_b x$ with $0 < b < 1$
0.4.6 Which of the following is a graph of \( y = \log_2 x \)?

0.4.7 Which of the following is a graph of \( y = \log_{\frac{1}{2}} x \)?
0.4.8 Which of the following functions have vertical asymptotes of $x = 3$?

(a) $y = \ln(x/3)$
(b) $y = \ln(x - 3)$
(c) $y = \ln(x + 3)$
(d) $y = 3 \ln x$

0.4.9 $\log \left( \frac{M-N}{M+N} \right) =$

(a) $2 \log M$
(b) $2 \log N$
(c) $-2 \log N$
0.4. LOGARITHMIC FUNCTIONS

(d) \( \log(M - N) - \log(M + N) \)

0.4.10 If \( \log_{10}(x - a) = n \), then \( x = \)

(a) \( 10^{n+a} \)
(b) \( a + 10^n \)
(c) \( n + 10^a \)
(d) \( n + a^{10} \)

0.4.11 What is the exponential form of \( \log_r m = j \)?

(a) \( r^j = m \)
(b) \( j^r = m \)
(c) \( m^j = r \)
(d) \( r^m = j \)

0.4.12 What is the logarithmic form of \( k^p = d \)?

(a) \( \log_d k = p \)
(b) \( \log_k d = p \)
(c) \( \log_p d = p \)
(d) \( \log_p d = p \)

0.4.13 What is the value of \( \log_{11} 86 \)? (Calculators are allowed.)

(a) \( .4049 \)
(b) \( .5383 \)
(c) \( 1.8576 \)
(d) \( -2.0564 \)

0.4.14 What is \( 3 = \log_2 8 \) in exponential form?

(a) \( 2^8 = 3 \)
(b) \( 3^2 = 8 \)
(c) \( 8^3 = 2 \)
(d) \(2^3 = 8\)

0.4.15 What is \(k = \log_m q\) in exponential form?

(a) \(m^k = q\)
(b) \(k^q = m\)
(c) \(m^q = k\)
(d) \(q^m = k\)

0.4.16 What is \(4^2 = 16\) in logarithmic form?

(a) \(\log_2 4 = 16\)
(b) \(\log_4 16 = 2\)
(c) \(\log_4 2 = 16\)
(d) \(\log_{16} 4 = 2\)

0.4.17 What is \(3^{-1} = \frac{1}{3}\) in logarithmic form?

(a) \(\log_3 (-1) = \frac{1}{3}\)
(b) \(\log_{-1} \frac{1}{3} = 3\)
(c) \(\log_{\frac{1}{3}} 3 = -1\)
(d) \(\log_3 \frac{1}{3} = -1\)

0.4.18 What is the inverse of the following function:

\[ P = f(t) = 16 \ln(14t) \]

(a) \(f^{-1}(P) = \frac{1}{14} e^{16P}\)
(b) \(f^{-1}(P) = \frac{1}{14} e^{P/16}\)
(c) \(f^{-1}(P) = \frac{1}{14} \ln(P/16)\)
(d) \(f^{-1}(P) = \frac{\ln 16}{14} P\)
0.4. LOGARITHMIC FUNCTIONS

0.4.19 Solve for $x$ if $8y = 3e^x$.

(a) $x = \ln 8 + \ln 3 + \ln y$
(b) $x = \ln 3 - \ln 8 + \ln y$
(c) $x = \ln 8 + \ln y - \ln 3$
(d) $x = \ln 3 - \ln 8 - \ln y$

0.4.20 Solve for $x$ if $y = e + 2^x$

(a) $x = \frac{\ln(y-1)}{\ln 2}$
(b) $x = \frac{\ln(y-1)}{\ln 2}$
(c) $x = \frac{\ln y}{\ln 2} - 1$
(d) $x = \frac{\ln(y-e)}{\ln 2}$

0.4.21 Write the following expression using a single logarithmic function:

$$\ln(2x^3 + 1) + 5 \ln(3 - x) - \ln(6x^5 + 2x + 1).$$

(a) $\ln((-6x^5 + 2x^3 - 7x + 15)$
(b) $\ln\left[(2x^3 + 1)(15 - 5x)(-6x^5 - 2x - 1]\right$
(c) $\ln\left(\frac{(2x^3 + 1)(3 - x)}{6x^5 + 2x + 1}\right$
(d) $\ln\left(\frac{(2x^3 + 1)(15 - 5x)}{6x^5 + 2x + 1}\right$

0.4.22 $\log\left(\frac{a^4b^7}{e^5}\right) =$

(a) $\log(a^4) + \log(b^7) + \log(e^5)$
(b) $4 \log a + 7 \log b - 5 \log c$
(c) $28 \log ab - 5 \log c$
(d) $\frac{28}{5} (\log a + \log b - \log c)$
(e) None of the above

0.4.23 Simplify the following expression: $\ln\left(\frac{\sqrt{x^2 + 1}(x^3 - 4)}{(3x - 7)^2}\right)$. 

(a) \( \frac{1}{2} \ln(x^2 + 1) + \ln(x^3 + 4) - 2 \ln(3x - 7) \)

(b) \( \ln \left( \frac{1}{2} (x^2 + 1) \right) + \ln(x^3 + 4) - 2 \ln(3x - 7) \)

(c) \( \ln(x^2 + 1) \ln(x^3 + 4) \ln(3x - 7) \)

(d) \( \ln[(x^2 + 1)(x^3 + 4)(3x - 7)] \)

0.4.24 25 rabbits are introduced to an island, where they quickly reproduce and the rabbit population grows according to an exponential model \( P(t) = P_0 e^{kt} \) so that the population doubles every four months. If \( t \) is in months, what is the value of the continuous growth rate \( k \)?

(a) \( k = \frac{1}{2} \ln 4 \)

(b) \( k = \frac{1}{4} \ln 2 \)

(c) \( k = \frac{1}{50} \ln \frac{4}{25} \)

(d) \( k = \frac{4}{25} \ln \frac{1}{50} \)

(e) None of the above

0.4.25 Simplify \( (\log_{16} 4) \left( \log_3 \frac{1}{9} \right) \).

(a) \( \frac{16}{3} \)

(b) \( \frac{4}{9} \)

(c) 1

(d) -1
0.5. TRIGONOMETRIC FUNCTIONS

0.5 Trigonometric Functions

Preview Activity 0.5. A tall water tower is swaying back and forth in the wind. Using an ultrasonic ranging device, we measure the distance from our device to the tower (in centimeters) every two seconds with these measurements recorded below.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (cm)</td>
<td>30.9</td>
<td>23.1</td>
<td>14.7</td>
<td>12.3</td>
<td>17.7</td>
<td>26.7</td>
<td>32.3</td>
<td>30.1</td>
<td>21.8</td>
<td>13.9</td>
<td>12.6</td>
</tr>
</tbody>
</table>

(a) Use the coordinate plane below to create a graph of these data points.

(b) What is the water tower’s maximum distance away from the device?

(c) What is the smallest distance measured from the tower to the device?

(d) If the water tower was sitting still and no wind was blowing, what would be the distance from the tower to the device? We call this the tower’s equilibrium position.

(e) What is the maximum distance that the tower moves away from its equilibrium position? We call this the amplitude of the oscillations.

(f) How much time does it take the water tower to sway back and forth in a complete cycle? We call this the period of oscillation.
Activity 0.16.

In this activity we will review the trigonometry of the special angles 0°, 30°, 45°, and their multiples.

(a) Use the fact that 180° is the same as π radians, convert each of the following angle measurements to radians.

<table>
<thead>
<tr>
<th>Degrees</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td>0</td>
<td>π/6</td>
<td>π/4</td>
<td>π/3</td>
<td>π/2</td>
<td>2π/3</td>
<td>3π/4</td>
<td>5π/6</td>
<td>π</td>
</tr>
</tbody>
</table>

(b) In part (a) of this problem there are several patterns that can help in remembering the radian conversions for certain angles. For example, you should have found that 30° converts to π/6 radians. Therefore, 60° should be twice π/6 which indeed it is: 60° = π/3 radians. What other similar patterns can you find? What is the minimum number of radian measures that you need to memorize?

(c) The sides of a 30 – 60 – 90 triangle follow well-known ratios. Consider the equilateral triangle on the left of the figure below. Fill in the rest of the sides and angles on the figure and use them to determine the trigonometric values of 30° and 60°.

(d) The sides of a 45 – 45 – 90 triangle also follow well-known ratios. Consider the isosceles triangle on the right of the figure below. Fill in the rest of the sides and angles on the figure and use them to determine the trigonometric values of 45°.

(e) Finally, we can organize all of the information about the special right triangles on a well-known organizational tool: the unit circle. The x coordinate of each point is the cosine of the angle and the y coordinate of each point is the sine of the angle.
Activity 0.17.

Figure 3 gives us the voltage produced by an electrical circuit as a function of time.

(a) What is the amplitude of the oscillations?
(b) What is the period of the oscillations?
(c) What is the average value of the voltage?
(d) What is the shift along the $t$ axis, $t_0$?
(e) What is a formula for this function?
**Activity 0.18.**

Suppose the following sinusoidal function models the water level on a pier in the ocean as it changes due to the tides during a certain day.

\[ w(t) = 4.3 \sin (0.51t + 0.82) + 10.6 \]

(a) Using the formula above, make a table showing the water level every two hours for a 24 hour period starting at midnight.

<table>
<thead>
<tr>
<th>time (hours)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>22</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>water level (ft)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Sketch a graph of this function using the data from your table in part (a).

(c) What is the period of oscillation of this function?

(d) What time is high tide?
Voting Questions

0.5.1 Which of the following is the approximate value for the sine and cosine of angles $A$ and $B$ in the figure below.

(a) $\sin A \approx 0.5, \cos A \approx 0.85, \sin B \approx -0.7, \cos B \approx 0.7$
(b) $\sin A \approx 0.85, \cos A \approx 0.5, \sin B \approx -0.7, \cos B \approx 0.7$
(c) $\sin A \approx 0.5, \cos A \approx 0.85, \sin B \approx 0.7, \cos B \approx 0.7$
(d) $\sin A \approx 0.85, \cos A \approx 0.5, \sin B \approx 0.7, \cos B \approx 0.7$

0.5.2 The amplitude and period of the function below are

(a) Amplitude = 2, Period = 2
(b) Amplitude = 2, Period = 3
(c) Amplitude = 2, Period = 1/2
(d) Amplitude = 3, Period = 2
(e) Amplitude = 3, Period = 1/2

0.5.3 What is the equation of the function shown in the graph?
0.5. TRIGONOMETRIC FUNCTIONS

(a) \( y = 3 \sin(2x) + 2 \)
(b) \( y = 3 \cos(2x) + 2 \)
(c) \( y = 3 \sin(\pi x) + 2 \)
(d) \( y = 3 \cos(\pi x) + 2 \)
(e) \( y = 3 \sin\left(\frac{1}{\pi} x\right) + 2 \)
(f) \( y = 3 \cos\left(\frac{1}{\pi} x\right) + 2 \)

0.5.4 The amplitude and period of the function below are

(a) Amplitude = 2, Period = 2
(b) Amplitude = 2, Period = 3
(c) Amplitude = 2, Period = 1/2
(d) Amplitude = 3, Period = 2
(e) Amplitude = 3, Period = 1/2

0.5.5 Which of the following could describe the graph below?

(a) \( y = 3 \cos(2x) \)
(b) \( y = 3 \cos\left(\frac{x}{2}\right) \)
(c) \( y = 3 \sin(2x) \)
(d) \( y = 3 \sin\left(\frac{x}{2}\right) \)
0.5.6 The function \( f(x) = 3 \sin(2x + 4) \) is created when you take the function \( g(x) = 3 \sin(2x) \) and you...

(a) shift it left by 4 units.
(b) shift it right by 4 units.
(c) shift it left by 2 units.
(d) shift it right by 2 units.
(e) shift it left by 8 units.

0.5.7 Which of the following could describe the graph below?

(a) \( y = 4 \sin \left( \pi x - \frac{\pi}{2} \right) - 2 \)
(b) \( y = -4 \sin \left( \pi x + \frac{\pi}{2} \right) - 2 \)
(c) \( y = -4 \cos(\pi x) - 2 \)
(d) \( y = 4 \cos(\pi(x + 1)) - 2 \)
(e) All of the above
(f) More than one, but not all of the above

0.5.8 What is an equation of the function whose graph is given below?

(a) \( f(x) = \cot x \)
(b) \( f(x) = \cot 2x \)
(c) \( f(x) = \cot \left( x - \frac{\pi}{2} \right) \)
(d) \( f(x) = \cot \left( 2x - \frac{\pi}{2} \right) \)
0.5.9 Three different functions of the form \( y = A \sin(Bx + C) \) are plotted below. Could these all have the same value of \( B \)?

(a) Yes  
(b) No  
(c) Not enough information is given.

0.5.10 The functions plotted below are all of the form \( y = A \sin(Bx + C) \). Which function has the largest value of \( B \)?
0.5.11 What is the phase shift of $f(x) = \frac{1}{5} \tan \left(2x + \frac{\pi}{2}\right)$?

(a) $2\pi$
(b) $\pi$
(c) $\frac{\pi}{2}$
(d) $\frac{\pi}{4}$
(e) $-2\pi$
(f) $-\pi$
(g) $-\frac{\pi}{2}$
(h) $-\frac{\pi}{4}$

0.5.12 What is the amplitude of $f(x) = -3 \sin(2x)$?

(a) 3
(b) -3
(c) $\pi$
(d) $2\pi$

0.5.13 What is the amplitude of $f(x) = -2 \sin x$?

(a) 1
(b) 2
(c) \(-2\)

0.5.14 What is the period of \(f(x) = -3\sin(2x)\)?

(a) 3
(b) -3
(c) \(\pi\)
(d) \(2\pi\)

0.5.15 What is the period of \(f(x) = \frac{1}{5}\tan(2x)\)?

(a) \(\frac{1}{5}\)
(b) \(2\pi\)
(c) \(\pi\)
(d) \(\frac{\pi}{2}\)
(e) \(\frac{\pi}{4}\)

0.5.16 Which of the basic trig functions below are odd functions?

(a) \(f(x) = \sin(x)\).
(b) \(f(x) = \cos(x)\).
(c) \(f(x) = \tan(x)\).
(d) (a) and (b).
(e) (a) and (c).
(f) (b) and (c).
(g) (a), (b), and (c).
(h) None of the above.
0.6 Powers, Polynomials, and Rational Functions

Preview Activity 0.6. Figure 4 shows the graphs of two different functions. Suppose that you were to graph a line anywhere along each of the two graphs.

1. Is it possible to draw a line that does not intersect the graph of \( f \)? \( g \)?

2. Is it possible to draw a line that intersects the graph of \( f \) an even number of times?

3. Is it possible to draw a line that intersects the graph of \( g \) an odd number of times?

4. What is the fewest number of intersections that your line could have with the graph of \( f \)? with \( g \)?

5. What is the largest number of intersections that your line could have with the graph of \( f \)? with \( g \)?

6. How many times does the graph of \( f \) change directions? How many times does the graph of \( g \) change directions?

![Figure 4: \( f(x) \) and \( g(x) \) for the preview activity.](image-url)
Activity 0.19.

Power functions and exponential functions appear somewhat similar in their formulas, but behave differently in many ways.

(a) Compare the functions $f(x) = x^2$ and $g(x) = 2^x$ by graphing both functions in several viewing windows. Find the points of intersection of the graphs. Which function grows more rapidly when $x$ is large?

(b) Compare the functions $f(x) = x^{10}$ and $g(x) = 2^x$ by graphing both functions in several viewing windows. Find the points of intersection of the graphs. Which function grows more rapidly when $x$ is large?

(c) Make a conjecture: As $x \to \infty$, which dominates, $x^n$ or $a^x$ for $a > 1$?
Activity 0.20.

For each of the following graphs, find a possible formula for the polynomial of lowest degree that fits the graph.
Activity 0.21.

(a) Suppose \( f(x) = x^2 + 3x + 2 \) and \( g(x) = x - 3 \).

(i) What is the behavior of the function \( h(x) = \frac{f(x)}{g(x)} \) near \( x = -1 \)? (i.e. what happens to \( h(x) \) as \( x \to -1 \)) near \( x = -2 \) near \( x = 3 \)?

(ii) What is the behavior of the function \( k(x) = \frac{g(x)}{f(x)} \) near \( x = -1 \) near \( x = -2 \) near \( x = 3 \)?

(b) Suppose \( f(x) = x^2 - 9 \) and \( g(x) = x - 3 \).

(i) What is the behavior of the function \( h(x) = \frac{f(x)}{g(x)} \) near \( x = -3 \)? (i.e. what happens to \( h(x) \) as \( x \to -3 \)) near \( x = 3 \)?

(ii) What is the behavior of the function \( k(x) = \frac{g(x)}{f(x)} \) near \( x = -3 \) near \( x = 3 \)?
Activity 0.22.

(a) Suppose $f(x) = x^3 + 2x^2 - x + 7$ and $g(x) = x^2 + 4x + 2$.

(i) Which function dominates as $x \to \infty$?

(ii) What is the behavior of the function $h(x) = \frac{f(x)}{g(x)}$ as $x \to \infty$?

(iii) What is the behavior of the function $k(x) = \frac{g(x)}{f(x)}$ as $x \to \infty$?

(b) Suppose $f(x) = 2x^4 - 5x^3 + 8x^2 - 3x - 1$ and $g(x) = 3x^4 - 2x^2 + 1$.

(i) Which function dominates as $x \to \infty$?

(ii) What is the behavior of the function $h(x) = \frac{f(x)}{g(x)}$ as $x \to \infty$?

(iii) What is the behavior of the function $k(x) = \frac{g(x)}{f(x)}$ as $x \to \infty$?

(c) Suppose $f(x) = e^x$ and $g(x) = x^{10}$.

(i) Which function dominates as $x \to \infty$ as $x \to \infty$?

(ii) What is the behavior of the function $h(x) = \frac{f(x)}{g(x)}$ as $x \to \infty$?

(iii) What is the behavior of the function $k(x) = \frac{g(x)}{f(x)}$ as $x \to \infty$?
Activity 0.23.

For each of the following functions, determine (1) whether the function has a horizontal asymptote, and (2) whether the function crosses its horizontal asymptote.

(a) \( f(x) = \frac{x + 3}{5x - 2} \)

(b) \( g(x) = \frac{x^2 + 2x - 1}{x - 1} \)

(c) \( h(x) = \frac{x + 1}{x^2 + 2x - 1} \)
Voting Questions

0.6.1 Which of the following is not a power function?
   
   (a) \( f(x) = 3x^2 \)
   (b) \( f(x) = x^{1.5} \)
   (c) \( f(x) = 6 \cdot 2^x \)
   (d) \( f(x) = -3x^{-\pi} \)

0.6.2 As \( x \to \infty \), which function dominates? That is, which function is larger in the long run?
   
   (a) \( 0.1x^2 \)
   (b) \( 10^{10}x \)

0.6.3 As \( x \to \infty \), which function dominates?
   
   (a) \( 0.25\sqrt{x} \)
   (b) \( 25,000x^{-3} \)

0.6.4 As \( x \to \infty \), which function dominates?
   
   (a) \( 3 - 0.9^x \)
   (b) \( \log x \)

0.6.5 Which function dominates as \( x \to \infty \)?
   
   (a) \( x^2 \)
   (b) \( e^x \)

0.6.6 As \( x \to \infty \), which function dominates?
   
   (a) \( x^3 \)
   (b) \( 2^x \)

0.6.7 As \( x \to \infty \), which function dominates?
0.6.8 Which of these functions dominates as \( x \to \infty \)?

(a) \( f(x) = -5x \)
(b) \( g(x) = 10^x \)
(c) \( h(x) = 0.9^x \)
(d) \( k(x) = x^5 \)
(e) \( l(x) = \pi^x \)

0.6.9 If \( f(x) = ax^2 + bx + c \) is a quadratic function, then the lowest point on the graph of \( f(x) \) occurs at \( x = -b/2a \).

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

0.6.10 Under what condition is the graph of the quadratic function described by \( f(x) = ax^2 + bx + c \) concave down?

(a) \( a < 0 \).
(b) \( b < 0 \).
(c) \( c < 0 \).
(d) More than one of the above.
(e) None of the above.

0.6.11 What is the degree of the graph of the polynomial in the figure below?
0.6.12 Which of the options below describes a function which is even?

(a) Any polynomial of even degree.
(b) Any polynomial of odd degree.
(c) \( f(x) = 9x^6 - 3x^2 + 2 \).
(d) \( f(x) = 3x^4 - 2x^3 + x^2 \).
(e) More than 1 of the above.
(f) None of the above.

0.6.13 The equation \( y = x^3 + 2x^2 - 5x - 6 \) is represented by which graph?
0.6.14 The graph below is a representation of which function?

(a) \( y = 3x + 2 \)
(b) \( y = (x - 2)(x + 3) \)
(c) \( y = (x - 6)(x - 2) \)
(d) \( y = (x - 3)(x + 2) \)
(e) none of these

0.6.15 Let \( f(x) = \frac{2x-1}{x+1} \) and \( g(x) = x - 1 \), then \( f(x) = g(x) \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

0.6.16 Which if the following is a graph for \( y = \frac{1-x^2}{x-2} \). (No calculators allowed.)
0.6.17 Which of the graphs represents $y = \frac{2x}{x-2}$?
Chapter 1

Understanding the Derivative

1.1 How do we measure velocity?

Preview Activity 1.1. This is the first Preview Activity in this text. Your job for this activity is to get to know the textbook.

(a) Where can you find the full textbook?

(b) What chapters of this text are you going to cover this semester. Have a look at your syllabus!

(c) What are the differences between Preview Activities, Activities, Examples, Exercises, Voting Questions, and WeBWork? Which ones should you do before class, which ones will you likely do during class, and which ones should you be doing after class?

(d) What materials in this text would you use to prepare for an exam and where do you find them?

(e) What should you bring to class every day?

▷
Preview Activity 1.2. Suppose that the height $s$ of a ball (in feet) at time $t$ (in seconds) is given by the formula $s(t) = 64 - 16(t - 1)^2$.

(a) Construct an accurate graph of $y = s(t)$ on the time interval $0 \leq t \leq 3$. Label at least six distinct points on the graph, including the three points that correspond to when the ball was released, when the ball reaches its highest point, and when the ball lands.

(b) In everyday language, describe the behavior of the ball on the time interval $0 < t < 1$ and on time interval $1 < t < 3$. What occurs at the instant $t = 1$?

(c) Consider the expression

$$AV_{[0.5,1]} = \frac{s(1) - s(0.5)}{1 - 0.5}.$$

Compute the value of $AV_{[0.5,1]}$. What does this value measure geometrically? What does this value measure physically? In particular, what are the units on $AV_{[0.5,1]}$?
Activity 1.1.

The following questions concern the position function given by \( s(t) = 64 - 16(t-1)^2 \), which is the same function considered in Preview Activity 1.2.

(a) Compute the average velocity of the ball on each of the following time intervals: \([0.4, 0.8], [0.7, 0.8], [0.79, 0.8], [0.799, 0.8], [0.8, 1.2], [0.8, 0.9], [0.8, 0.81], [0.8, 0.801]\). Include units for each value.

(b) On the provided graph in Figure 1.1, sketch the line that passes through the points \( A = (0.4, s(0.4)) \) and \( B = (0.8, s(0.8)) \). What is the meaning of the slope of this line? In light of this meaning, what is a geometric way to interpret each of the values computed in the preceding question?

(c) Use a graphing utility to plot the graph of \( s(t) = 64 - 16(t-1)^2 \) on an interval containing the value \( t = 0.8 \). Then, zoom in repeatedly on the point \((0.8, s(0.8))\). What do you observe about how the graph appears as you view it more and more closely?

(d) What do you conjecture is the velocity of the ball at the instant \( t = 0.8 \)? Why?

Figure 1.1: A partial plot of \( s(t) = 64 - 16(t-1)^2 \).
Activity 1.2.

Each of the following questions concern $s(t) = 64 - 16(t - 1)^2$, the position function from Preview Activity 1.2.

(a) Compute the average velocity of the ball on the time interval $[1.5, 2]$. What is different between this value and the average velocity on the interval $[0, 0.5]$?

(b) Use appropriate computing technology to estimate the instantaneous velocity of the ball at $t = 1.5$. Likewise, estimate the instantaneous velocity of the ball at $t = 2$. Which value is greater?

(c) How is the sign of the instantaneous velocity of the ball related to its behavior at a given point in time? That is, what does positive instantaneous velocity tell you the ball is doing? Negative instantaneous velocity?

(d) Without doing any computations, what do you expect to be the instantaneous velocity of the ball at $t = 1$? Why?
Activity 1.3.

For the function given by \( s(t) = 64 - 16(t - 1)^2 \) from Preview Activity 1.2, find the most simplified expression you can for the average velocity of the ball on the interval \([2, 2 + h]\). Use your result to compute the average velocity on \([1.5, 2]\) and to estimate the instantaneous velocity at \( t = 2 \). Finally, compare your earlier work in Activity 1.1.
Voting Questions

1.1.1 The speedometer in my car is broken. In order to find my average velocity on a trip from Helena to Missoula, I need
   i. the distance between Helena and Missoula
   ii. the time spent traveling
   iii. the number of stops I made during the trip
   iv. a friend with a stopwatch
   v. a working odometer
   vi. none of the above

Select the best combination:

(a) i, ii, & iii only
(b) i & ii only
(c) iv & v only
(d) vi
(e) a combination that is not listed here

1.1.2 The speedometer in my car is broken. In order to find my velocity at the instant I hit a speed trap, I need
   i. the distance between Helena and Missoula
   ii. the time spent traveling
   iii. the number of stops I made during the trip
   iv. a friend with a stopwatch
   v. a working odometer
   vi. none of the above

Select the best combination:

(a) i, ii, & iii only
(b) i & ii only
(c) iv & v only
(d) vi
(e) a combination that is not listed here
1.1. HOW DO WE MEASURE VELOCITY?

1.1.3 Which graph represents an object slowing down, where \( D \) is distance, and \( t \) is time? Assume that the units are the same for all graphs.

1.1.4 True or False: If a car is going 50 miles per hour at 2 pm and 60 miles per hour at 3 pm, then it travels between 50 and 60 miles during the hour between 2 pm and 3 pm.

   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

1.1.5 True or False: If a car travels 80 miles between 2 and 4 pm, then its velocity is close to 40 mph at 2 pm.

   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

1.1.6 True or False: If the time interval is short enough, then the average velocity of a car over the time interval and the instantaneous velocity at a time in the interval can be expected to be close.

   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident
1.1.7 True or False: If an object moves with the same average velocity over every time interval, then its average velocity equals its instantaneous velocity at any time.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
1.2 The notion of limit

Preview Activity 1.3. Suppose that \( g \) is the function given by the graph below. Use the graph to answer each of the following questions.

(a) Determine the values \( g(-2) \), \( g(-1) \), \( g(0) \), \( g(1) \), and \( g(2) \), if defined. If the function value is not defined, explain what feature of the graph tells you this.

(b) For each of the values \( a = -1 \), \( a = 0 \), and \( a = 2 \), complete the following sentence: “As \( x \) gets closer and closer (but not equal) to \( a \), \( g(x) \) gets as close as we want to \hphantom{a}.”

(c) What happens as \( x \) gets closer and closer (but not equal) to \( a = 1 \)? Does the function \( g(x) \) get as close as we would like to a single value?

Figure 1.2: Graph of \( y = g(x) \) for Preview Activity 1.3.
Activity 1.4.

Estimate the value of each of the following limits by constructing appropriate tables of values. Then determine the exact value of the limit by using algebra to simplify the function. Finally, plot each function on an appropriate interval to check your result visually.

(a) \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \)

(b) \( \lim_{x \to 0} \frac{(2 + x)^3 - 8}{x} \)

(c) \( \lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} \)
Activity 1.5.

Consider a moving object whose position function is given by \( s(t) = t^2 \), where \( s \) is measured in meters and \( t \) is measured in minutes.

(a) Determine a simplified expression for the average velocity of the object on the interval \([3, 3 + h]\).

(b) Determine the average velocity of the object on the interval \([3, 3.2]\). Include units on your answer.

(c) Determine the instantaneous velocity of the object when \( t = 3 \). Include units on your answer.
Activity 1.6.

For the moving object whose position $s$ at time $t$ is given by the graph below, answer each of the following questions. Assume that $s$ is measured in feet and $t$ is measured in seconds.

![Graph of position function $y = s(t)$](image)

Figure 1.3: Plot of the position function $y = s(t)$ in Activity 1.6.

(a) Use the graph to estimate the average velocity of the object on each of the following intervals: $[0.5, 1]$, $[1.5, 2.5]$, $[0, 5]$. Draw each line whose slope represents the average velocity you seek.

(b) How could you use average velocities and slopes of lines to estimate the instantaneous velocity of the object at a fixed time?

(c) Use the graph to estimate the instantaneous velocity of the object when $t = 2$. Should this instantaneous velocity at $t = 2$ be greater or less than the average velocity on $[1.5, 2.5]$ that you computed in (a)? Why?
1.2. THE NOTION OF LIMIT

Voting Questions

1.2.1 Consider the function:

\[ f(x) = \begin{cases} 
6 & \text{if } x > 9 \\
2 & \text{if } x = 9 \\
-x + 14 & \text{if } -7 \leq x < 9 \\
21 & \text{if } x < -7 
\end{cases} \]

(a) \( \lim_{x \to 9^-} f(x) = 2 \)
(b) \( \lim_{x \to 9^-} f(x) = 5 \)
(c) \( \lim_{x \to 9^-} f(x) = 6 \)
(d) \( \lim_{x \to 9^-} f(x) = 14 \)
(e) \( \lim_{x \to 9^-} f(x) = 21 \)

1.2.2 True or False: As \( x \) increases to 100, \( f(x) = 1/x \) gets closer and closer to 0, so the limit as \( x \) goes to 100 of \( f(x) \) is 0. Be prepared to justify your answer.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.2.3 True or False: \( \lim_{x \to a} f(x) = L \) means that if \( x_1 \) is closer to \( a \) than \( x_2 \) is, then \( f(x_1) \) will be closer to \( L \) than \( f(x_2) \) is. Be prepared to justify your answer with an argument or counterexample.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.2.4 The reason that \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) does not exist is:

(a) because no matter how close \( x \) gets to 0, there are \( x \)'s near 0 for which \( \sin \left( \frac{1}{x} \right) = 1 \), and some for which \( \sin \left( \frac{1}{x} \right) = -1 \).
(b) because the function values oscillate around 0.
(c) because $\frac{1}{0}$ is undefined.
(d) all of the above

1.2.5 $\lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right)$

(a) does not exist because no matter how close $x$ gets to 0, there are $x$’s near 0 for which $\sin \left( \frac{1}{x} \right) = 1$, and some for which $\sin \left( \frac{1}{x} \right) = -1$.
(b) does not exist because the function values oscillate around 0.
(c) does not exist because $\frac{1}{0}$ is undefined.
(d) equals 0
(e) equals 1

1.2.6 You’re trying to guess $\lim_{x \to 0} f(x)$. You plug in $x = 0.1, 0.01, 0.001, \cdots$ and get $f(x) = 0$ for all of these values. In fact you’re told that for all $n = 1, 2, \cdots$, $f \left( \frac{1}{10^n} \right) = 0$. True or False: Since the sequence $f(0.1), f(0.01), f(0.001), \cdots$ goes to 0, we know that $\lim_{x \to 0} = 0$.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.2.7 If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$

(a) does not exist.
(b) must exist.
(c) can’t be determined. Not enough information is given.

1.2.8 True or False: Consider a function $f(x)$ with the property that $\lim_{x \to a} f(x) = 0$. Now consider another function $g(x)$ also defined near $a$. Then $\lim_{x \to a} [f(x)g(x)] = 0$.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
1.2.9 If a function $f$ is not defined at $x = a$,

(a) $\lim_{x \to a}$ cannot exist.
(b) $\lim_{x \to a}$ could be 0.
(c) $\lim_{x \to a}$ must approach $\infty$.
(d) none of the above

1.2.10 Possible criteria for continuity at a point: If the limit of the function exists at a point, the function is continuous at that point. Which of the following examples fits the above criteria but is not continuous at $x = 0$?

(a) $f(x) = x$
(b) $f(x) = x^2/x$
(c) $f(x) = |x|/x$
(d) None of these show a problem with this criteria.

1.2.11 Let $f(x) = 5x^4 + 18x^3 - 2x + 3$. As $x$ gets really big, what becomes the most important (dominant) term in this function?

(a) $5x^4$
(b) $18x^3$
(c) $-2x$
(d) 3

1.2.12 What is

$$\lim_{x \to \infty} \frac{6x^2 - 5x}{2x^2 + 3}?$$

(a) 0
(b) 2
(c) 3
(d) 6
(e) infinity

1.2.13 What is

$$\lim_{x \to \infty} \frac{3x^2 + 5x^3 - 2x + 4}{4x^3 - 5x + 6}?$$
1.2.14 What is \( \lim_{x \to \infty} \frac{100x^5 - 15x}{x^6 + 3} \)?

(a) 0  
(b) 5/6  
(c) 85  
(d) 100  
(e) infinity

1.2.15 What is \( \lim_{x \to \infty} \frac{x^2 + 2x + 3}{25x - 7} \)?

(a) 0  
(b) 1/25  
(c) 3/7  
(d) 2  
(e) infinity

1.2.16 Let \( f(x) = \frac{x^2 - 4x + 3}{x^2 - 1} \). Evaluate \( \lim_{x \to -1} f(x) \).

(a) \(-1\)  
(b) \(\infty\)  
(c) \(-\infty\)
1.3 The derivative of a function at a point

Preview Activity 1.4. Suppose that $f$ is the function given by the graph below and that $a$ and $a + h$ are the input values as labeled on the $x$-axis. Use the graph in Figure 1.4 to answer the following questions.

(a) Locate and label the points $(a, f(a))$ and $(a + h, f(a + h))$ on the graph.

(b) Construct a right triangle whose hypotenuse is the line segment from $(a, f(a))$ to $(a + h, f(a + h))$. What are the lengths of the respective legs of this triangle?

(c) What is the slope of the line that connects the points $(a, f(a))$ and $(a + h, f(a + h))$?

(d) Write a meaningful sentence that explains how the average rate of change of the function on a given interval and the slope of a related line are connected.

Figure 1.4: Plot of $y = f(x)$ for Preview Activity 1.4.
Activity 1.7.

Consider the function \( f \) whose formula is \( f(x) = 3 - 2x \).

(a) What familiar type of function is \( f \)? What can you say about the slope of \( f \) at every value of \( x \)?

(b) Compute the average rate of change of \( f \) on the intervals \([1, 4]\), \([3, 7]\), and \([5, 5 + h]\); simplify each result as much as possible. What do you notice about these quantities?

(c) Use the limit definition of the derivative to compute the exact instantaneous rate of change of \( f \) with respect to \( x \) at the value \( a = 1 \). That is, compute \( f'(1) \) using the limit definition. Show your work. Is your result surprising?

(d) Without doing any additional computations, what are the values of \( f'(2) \), \( f'(\pi) \), and \( f'(-\sqrt{2}) \)? Why?
Activity 1.8.

A water balloon is tossed vertically in the air from a window. The balloon’s height in feet at time \( t \) in seconds after being launched is given by \( s(t) = -16t^2 + 16t + 32 \). Use this function to respond to each of the following questions.

(a) Sketch an accurate, labeled graph of \( s \) on the axes provided in Figure 1.5. You should be able to do this without using computing technology.

(b) Compute the average rate of change of \( s \) on the time interval \([1, 2]\). Include units on your answer and write one sentence to explain the meaning of the value you found.

(c) Use the limit definition to compute the instantaneous rate of change of \( s \) with respect to time, \( t \), at the instant \( a = 1 \). Show your work using proper notation, include units on your answer, and write one sentence to explain the meaning of the value you found.

(d) On your graph in (a), sketch two lines: one whose slope represents the average rate of change of \( s \) on \([1, 2]\), the other whose slope represents the instantaneous rate of change of \( s \) at the instant \( a = 1 \). Label each line clearly.

(e) For what values of \( a \) do you expect \( s'(a) \) to be positive? Why? Answer the same questions when “positive” is replaced by “negative” and “zero.”
Activity 1.9.

A rapidly growing city in Arizona has its population $P$ at time $t$, where $t$ is the number of decades after the year 2010, modeled by the formula $P(t) = 25000e^{t/5}$. Use this function to respond to the following questions.

(a) Sketch an accurate graph of $P$ for $t = 0$ to $t = 5$ on the axes provided in Figure 1.6. Label the scale on the axes carefully.

(b) Compute the average rate of change of $P$ between 2030 and 2050. Include units on your answer and write one sentence to explain the meaning (in everyday language) of the value you found.

(c) Use the limit definition to write an expression for the instantaneous rate of change of $P$ with respect to time, $t$, at the instant $a = 2$. Explain why this limit is difficult to evaluate exactly.

(d) Estimate the limit in (c) for the instantaneous rate of change of $P$ at the instant $a = 2$ by using several small $h$ values. Once you have determined an accurate estimate of $P'(2)$, include units on your answer, and write one sentence (using everyday language) to explain the meaning of the value you found.

(e) On your graph above, sketch two lines: one whose slope represents the average rate of change of $P$ on $[2, 4]$, the other whose slope represents the instantaneous rate of change of $P$ at the instant $a = 2$.

(f) In a carefully-worded sentence, describe the behavior of $P'(a)$ as $a$ increases in value. What does this reflect about the behavior of the given function $P$?\[\text{\textcopyright 2014 by John Wiley & Sons, Inc. All rights reserved.} \]
1.3. THE DERIVATIVE OF A FUNCTION AT A POINT

Voting Questions

1.3.1 We want to find how the volume of a balloon changes as it is filled with air. We know \( V(r) = \frac{4}{3} \pi r^3 \), where \( r \) is the radius in inches and \( V(r) \) is the volume in cubic inches. The expression \( \frac{V(3) - V(1)}{3 - 1} \) represents the

(a) Average rate of change of the radius with respect to the volume when the radius changes from 1 inch to 3 inches.
(b) Average rate of change of the radius with respect to the volume when the volume changes from 1 cubic inch to 3 cubic inches.
(c) Average rate of change of the volume with respect to the radius when the radius changes from 1 inch to 3 inches.
(d) Average rate of change of the volume with respect to the radius when the volume changes from 1 cubic inch to 3 cubic inches.

1.3.2 We want to find how the volume of a balloon changes as it is filled with air. We know \( V(r) = \frac{4}{3} \pi r^3 \), where \( r \) is the radius in inches and \( V(r) \) is the volume in cubic inches. Which of the following represents the rate at which the volume is changing when the radius is 1 inch?

(a) \( \frac{V(1.01) - V(1)}{0.01} \approx 12.69 \)
(b) \( \frac{V(0.99) - V(1)}{-0.01} \approx 12.44 \)
(c) \( \lim_{h \to 0} \frac{V(1+h) - V(1)}{h} \)
(d) All of the above

1.3.3 Which of the following represents the slope of a line drawn between the two points marked in the figure?

(a) \( m = \frac{F(a) + F(b)}{a + b} \)
(b) \( m = \frac{F(b) - F(a)}{b - a} \)
(c) \( m = \frac{a}{b} \)
(d) \( m = \frac{F(a) - F(b)}{b - a} \)
1.3.4 The line tangent to the graph of \( f(x) = x \) at (0,0)

(a) is \( y = 0 \)
(b) is \( y = x \)
(c) does not exist
(d) is not unique. There are infinitely many tangent lines.

1.3.5 Suppose that \( f(x) \) is a function with \( f(2) = 15 \) and \( f'(2) = 3 \). Estimate \( f(2.5) \).

(a) 10.5
(b) 15
(c) 16.5
(d) 18
1.4 The derivative function

Preview Activity 1.5. Consider the function \( f(x) = 4x - x^2 \).

(a) Use the limit definition to compute the following derivative values: \( f'(0) \), \( f'(1) \), \( f'(2) \), and \( f'(3) \).

(b) Observe that the work to find \( f'(a) \) is the same, regardless of the value of \( a \). Based on your work in (a), what do you conjecture is the value of \( f'(4) \)? How about \( f'(5) \)? (Note: you should not use the limit definition of the derivative to find either value.)

(c) Conjecture a formula for \( f'(a) \) that depends only on the value \( a \). That is, in the same way that we have a formula for \( f(x) \) (recall \( f(x) = 4x - x^2 \)), see if you can use your work above to guess a formula for \( f'(a) \) in terms of \( a \).
Activity 1.10.

For each given graph of \( y = f(x) \), sketch an approximate graph of its derivative function, \( y = f'(x) \), on the axes immediately below. The scale of the grid for the graph of \( f \) is \( 1 \times 1 \); assume the horizontal scale of the grid for the graph of \( f' \) is identical to that for \( f \). If necessary, adjust and label the vertical scale on the axes for the graph of \( f' \).
Write several sentences that describe your overall process for sketching the graph of the derivative function, given the graph the original function. What are the values of the derivative function that you tend to identify first? What do you do thereafter? How do key traits of the graph of the derivative function exemplify properties of the graph of the original function?
Activity 1.11.

For each of the listed functions, determine a formula for the derivative function. For the first two, determine the formula for the derivative by thinking about the nature of the given function and its slope at various points; do not use the limit definition. For the latter four, use the limit definition. Pay careful attention to the function names and independent variables. It is important to be comfortable with using letters other than $f$ and $x$. For example, given a function $p(z)$, we call its derivative $p'(z)$.

(a) $f(x) = 1$
(b) $g(t) = t$
(c) $p(z) = z^2$
(d) $q(s) = s^3$
(e) $F(t) = \frac{1}{t}$
(f) $G(y) = \sqrt{y}$
Voting Questions

1.4.1 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.6?

1.4.2 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.8?

1.4.3 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.9?
1.4.4 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.10?

1.4.5 Which of the following graphs is the graph of the derivative of the function shown in Figure 2.11?
1.4.6 The graph in Figure 2.12 is the derivative of which of the following functions?

![Figure 2.12](image1)

1.4.7 The graph in Figure 2.13 is the derivative of which of the following functions?

![Figure 2.13](image2)

1.4.8 The graph in Figure 2.14 is the derivative of which of the following functions?
1.4.9 The graph in Figure 2.15 is the derivative of which of the following functions?

1.4.10 True or False: If \( f'(x) = g'(x) \) then \( f(x) = g(x) \).
(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.4.11 Let \( f(x) = 2x^3 + 3x^2 + 1 \). True or false: On the interval \((-\infty, -1)\), the function \( f \) is increasing.
(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.
1.5 Interpreting the derivative and its units

**Preview Activity 1.6.** One of the longest stretches of straight (and flat) road in North America can be found on the Great Plains in the state of North Dakota on state highway 46, which lies just south of the interstate highway I-94 and runs through the town of Gackle. A car leaves town (at time \( t = 0 \)) and heads east on highway 46; its position in miles from Gackle at time \( t \) in minutes is given by the graph of the function in Figure 1.7. Three important points are labeled on the graph; where the curve looks linear, assume that it is indeed a straight line.

![Graph of y = s(t)](image)

Figure 1.7: The graph of \( y = s(t) \), the position of the car along highway 46, which tells its distance in miles from Gackle, ND, at time \( t \) in minutes.

(a) In everyday language, describe the behavior of the car over the provided time interval. In particular, discuss what is happening on the time intervals \([57, 68]\) and \([68, 104]\).

(b) Find the slope of the line between the points \((57, 63.8)\) and \((104, 106.8)\). What are the units on this slope? What does the slope represent?

(c) Find the average rate of change of the car’s position on the interval \([68, 104]\). Include units on your answer.

(d) Estimate the instantaneous rate of change of the car’s position at the moment \( t = 80 \). Write a sentence to explain your reasoning and the meaning of this value.
Activity 1.12.

A potato is placed in an oven, and the potato’s temperature $F$ (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time $t$ is measured in minutes.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$F(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>70</td>
</tr>
<tr>
<td>15</td>
<td>180.5</td>
</tr>
<tr>
<td>30</td>
<td>251</td>
</tr>
<tr>
<td>45</td>
<td>296</td>
</tr>
<tr>
<td>60</td>
<td>324.5</td>
</tr>
<tr>
<td>75</td>
<td>342.8</td>
</tr>
<tr>
<td>90</td>
<td>354.5</td>
</tr>
</tbody>
</table>

(a) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t = 30$. Include units on your answer.

(b) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t = 60$. Include units on your answer.

(c) Without doing any calculation, which do you expect to be greater: $F'(75)$ or $F'(90)$? Why?

(d) Suppose it is given that $F(64) = 330.28$ and $F'(64) = 1.341$. What are the units on these two quantities? What do you expect the temperature of the potato to be when $t = 65$? when $t = 66$? Why?

(e) Write a couple of careful sentences that describe the behavior of the temperature of the potato on the time interval $[0, 90]$, as well as the behavior of the instantaneous rate of change of the temperature of the potato on the same time interval.
Activity 1.13.

A company manufactures rope, and the total cost of producing \( r \) feet of rope is \( C(r) \) dollars.

(a) What does it mean to say that \( C(2000) = 800 \)?

(b) What are the units of \( C'(r) \)?

(c) Suppose that \( C(2000) = 800 \) and \( C'(2000) = 0.35 \). Estimate \( C(2100) \), and justify your estimate by writing at least one sentence that explains your thinking.

(d) Which of the following statements do you think is true, and why?
   - \( C'(2000) < C'(3000) \)
   - \( C'(2000) = C'(3000) \)
   - \( C'(2000) > C'(3000) \)

(e) Suppose someone claims that \( C'(5000) = -0.1 \). What would the practical meaning of this derivative value tell you about the approximate cost of the next foot of rope? Is this possible? Why or why not?
Activity 1.14.

Researchers at a major car company have found a function that relates gasoline consumption to speed for a particular model of car. In particular, they have determined that the consumption $C$, in liters per kilometer, at a given speed $s$, is given by a function $C = f(s)$, where $s$ is the car’s speed in kilometers per hour.

(a) Data provided by the car company tells us that $f(80) = 0.015$, $f(90) = 0.02$, and $f(100) = 0.027$. Use this information to estimate the instantaneous rate of change of fuel consumption with respect to speed at $s = 90$. Be as accurate as possible, use proper notation, and include units on your answer.

(b) By writing a complete sentence, interpret the meaning (in the context of fuel consumption) of “$f(80) = 0.015$.”

(c) Write at least one complete sentence that interprets the meaning of the value of $f'(90)$ that you estimated in (a).
Voting Questions

1.5.1 The radius of a snowball changes as the snow melts. The instantaneous rate of change in radius with respect to volume is

(a) \( \frac{dV}{dr} \)
(b) \( \frac{dr}{dV} \)
(c) \( \frac{dV}{dr} + \frac{dr}{dV} \)
(d) None of the above

1.5.2 Gravel is poured into a conical pile. The rate at which gravel is added to the pile is

(a) \( \frac{dV}{dt} \)
(b) \( \frac{dr}{dt} \)
(c) \( \frac{dV}{dr} \)
(d) None of the above

1.5.3 A slow freight train chugs along a straight track. The distance it has traveled after \( x \) hours is given by a function \( f(x) \). An engineer is walking along the top of the box cars at the rate of 3 mi/hr in the same direction as the train is moving. The speed of the man relative to the ground is

(a) \( f(x) + 3 \)
(b) \( f'(x) + 3 \)
(c) \( f(x) - 3 \)
(d) \( f'(x) - 3 \)

1.5.4 \( C(r) \) gives the total cost of paying off a car loan that has an annual interest rate of \( r \) %. What are the units of \( C'(r) \)?

(a) Year / $
(b) $ / Year
(c) $ / %
(d) % / $
1.5.5 \( C(r) \) gives the total cost of paying off a car loan that has an annual interest rate of \( r \% \). What is the practical meaning of \( C'(5) \)?

(a) The rate of change of the total cost of the car loan is \( C'(5) \).
(b) If the interest rate increases by 1%, then the total cost of the loan increases by \( C'(5) \).
(c) If the interest rate increases by 1%, then the total cost of the loan increases by \( C'(5) \) when the interest rate is 5%.
(d) If the interest rate increases by 5%, then the total cost of the loan increases by \( C'(5) \).

1.5.6 \( C(r) \) gives the total cost of paying off a car loan that has an annual interest rate of \( r \% \). What is the sign of \( C'(5) \)?

(a) Positive
(b) Negative
(c) Not enough information is given

1.5.7 \( g(v) \) gives the fuel efficiency, in miles per gallon, of a car going a speed of \( v \) miles per hour. What are the units of \( g'(v) = \frac{dg}{dv} \)?

(a) (miles)\(^2\) / [(gal)(hour)]
(b) hour/gal
(c) gal/hour
(d) (gal)(hour)/(miles)\(^2\)

1.5.8 \( g(v) \) gives the fuel efficiency, in miles per gallon, of a car going a speed of \( v \) miles per hour. What is the practical meaning of \( g'(55) = -0.54 \)?

(a) When the car is going 55 mph, the rate of change of the fuel efficiency decreases to 0.54 miles/gal.
(b) When the car is going 55 mph, the rate of change of the fuel efficiency decreases by 0.54 miles/gal.
(c) If the car speeds up from 55 to 56 mph, then the fuel efficiency is 0.54 miles per gallon.
(d) If the car speeds up from 55 to 56 mph, then the car becomes less fuel efficient by 0.54 miles per gallon.
1.6 The second derivative

Preview Activity 1.7. The position of a car driving along a straight road at time \( t \) in minutes is given by the function \( y = s(t) \) that is pictured in Figure 1.8. The car’s position function has units measured in thousands of feet. For instance, the point \((2, 4)\) on the graph indicates that after 2 minutes, the car has traveled 4000 feet.

![Graph of \( y = s(t) \)](image)

Figure 1.8: The graph of \( y = s(t) \), the position of the car (measured in thousands of feet from its starting location) at time \( t \) in minutes.

(a) In everyday language, describe the behavior of the car over the provided time interval. In particular, you should carefully discuss what is happening on each of the time intervals \([0, 1]\), \([1, 2]\), \([2, 3]\), \([3, 4]\), and \([4, 5]\), plus provide commentary overall on what the car is doing on the interval \([0, 12]\).

(b) On the lefthand axes provided in Figure 1.9, sketch a careful, accurate graph of \( y = s'(t) \).

(c) What is the meaning of the function \( y = s'(t) \) in the context of the given problem? What can we say about the car’s behavior when \( s'(t) \) is positive? when \( s'(t) \) is zero? when \( s'(t) \) is negative?

(d) Rename the function you graphed in (b) to be called \( y = v(t) \). Describe the behavior of \( v \) in words, using phrases like “\( v \) is increasing on the interval . . .” and “\( v \) is constant on the interval . . .”

(e) Sketch a graph of the function \( y = v'(t) \) on the righthand axes provide in Figure 1.8. Write at least one sentence to explain how the behavior of \( v'(t) \) is connected to the graph of \( y = v(t) \).
Activity 1.15.

The position of a car driving along a straight road at time $t$ in minutes is given by the function $y = s(t)$ that is pictured in Figure 1.10. The car’s position function has units measured in thousands of feet. Remember that you worked with this function and sketched graphs of $y = v(t) = s'(t)$ and $y = v'(t)$ in Preview Activity 1.7.

(a) On what intervals is the position function $y = s(t)$ increasing? decreasing? Why?

(b) On which intervals is the velocity function $y = v(t) = s'(t)$ increasing? decreasing? neither? Why?

(c) Acceleration is defined to be the instantaneous rate of change of velocity, as the acceleration of an object measures the rate at which the velocity of the object is changing. Say that the car’s acceleration function is named $a(t)$. How is $a(t)$ computed from $v(t)$? How is $a(t)$ computed from $s(t)$? Explain.

(d) What can you say about $s''$ whenever $s'$ is increasing? Why?

(e) Using only the words increasing, decreasing, constant, concave up, concave down, and linear, complete the following sentences. For the position function $s$ with velocity $v$ and acceleration $a$,

- on an interval where $v$ is positive, $s$ is ____________________.
- on an interval where $v$ is negative, $s$ is ____________________.
- on an interval where $v$ is zero, $s$ is ____________________.
- on an interval where $a$ is positive, $v$ is ____________________.
- on an interval where $a$ is negative, $v$ is ____________________.
- on an interval where $a$ is zero, $v$ is ____________________.
Figure 1.10: The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time $t$ in minutes.

- on an interval where $a$ is positive, $s$ is ________________.
- on an interval where $a$ is negative, $s$ is ________________.
- on an interval where $a$ is zero, $s$ is ________________.
Activity 1.16.

This activity builds on our experience and understanding of how to sketch the graph of $f'$ given the graph of $f$. Below, given the graph of a function $f$, sketch $f'$ on the first axes below, and then sketch $f''$ on the second set of axes. In addition, for each, write several careful sentences in the spirit of those in Activity 1.15 that connect the behaviors of $f$, $f'$, and $f''$. For instance, write something such as

\[
\begin{align*}
  f' \text{ is } & \underline{\phantom{00}} \text{ on the interval } \underline{\phantom{00}}, \text{ which is connected to the fact that } \\
  f \text{ is } & \underline{\phantom{00}} \text{ on the same interval } \underline{\phantom{00}}, \text{ and } f'' \text{ is } \underline{\phantom{00}} \\
  \text{on the interval as well} 
\end{align*}
\]

but of course with the blanks filled in. Throughout, view the scale of the grid for the graph of $f$ as being $1 \times 1$, and assume the horizontal scale of the grid for the graph of $f'$ is identical to that for $f$. If you need to adjust the vertical scale on the axes for the graph of $f'$ or $f''$, you should label that accordingly.

\[\triangledown\]
Activity 1.17.

A potato is placed in an oven, and the potato’s temperature $F$ (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time $t$ is measured in minutes. In Activity 1.12, we computed approximations to $F'(30)$ and $F'(60)$ using central differences. Those values and more are provided in the second table below, along with several others computed in the same way.
1.6. THE SECOND DERIVATIVE

<table>
<thead>
<tr>
<th>$t$</th>
<th>$F(t)$</th>
<th>$t$</th>
<th>$F'(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>70</td>
<td>0</td>
<td>NA</td>
</tr>
<tr>
<td>15</td>
<td>180.5</td>
<td>15</td>
<td>6.03</td>
</tr>
<tr>
<td>30</td>
<td>251</td>
<td>30</td>
<td>3.85</td>
</tr>
<tr>
<td>45</td>
<td>296</td>
<td>45</td>
<td>2.45</td>
</tr>
<tr>
<td>60</td>
<td>324.5</td>
<td>60</td>
<td>1.56</td>
</tr>
<tr>
<td>75</td>
<td>342.8</td>
<td>75</td>
<td>1.00</td>
</tr>
<tr>
<td>90</td>
<td>354.5</td>
<td>90</td>
<td>NA</td>
</tr>
</tbody>
</table>

(a) What are the units on the values of $F'(t)$?

(b) Use a central difference to estimate the value of $F''(30)$.

(c) What is the meaning of the value of $F''(30)$ that you have computed in (c) in terms of the potato’s temperature? Write several careful sentences that discuss, with appropriate units, the values of $F(30)$, $F'(30)$, and $F''(30)$, and explain the overall behavior of the potato’s temperature at this point in time.

(d) Overall, is the potato’s temperature increasing at an increasing rate, increasing at a constant rate, or increasing at a decreasing rate? Why?
Voting Questions

1.6.1 The graph of \( y = f(x) \) is shown in figure 2.18. Which of the following is true for \( f \) on the interval shown?

- i. \( f(x) \) is positive
- ii. \( f(x) \) is increasing
- iii. \( f'(x) \) is positive
- iv. \( f''(x) \) is increasing
- v. \( f'''(x) \) is positive

(a) i, ii, and iii only
(b) ii, iii, and v only
(c) ii, iii, iv, and v only
(d) all are true
(e) the correct combination of true statements is not listed here

1.6.2 True or False: If \( f''(x) \) is positive, then \( f(x) \) is concave up.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.6.3 True or False: If \( f''(x) \) is positive, then \( f'(x) \) is increasing.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.6.4 True or False: If \( f'(x) \) is increasing, then \( f(x) \) is concave up.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.6.5 **True or False**: If the velocity of an object is constant, then its acceleration is zero.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.6.6 In Figure 2.21, the second derivative at points \( a \), \( b \), and \( c \), respectively, is

(a) +, 0, -
(b) -, 0, +
(c) -, 0, -
(d) +, 0, +
(e) +, +, -
(f) -, -, +

1.6.7 In figure 2.22, at \( x = 0 \) the signs of the function and the first and second derivatives, in order, are

(a) +, 0, +
(b) +, 0, -
(c) -, +, -
(d) -, +, +
(e) +, -, +
(f) +, +, +

1.6.8 Which of the following graphs could represent the second derivative of the function shown in Figure 2.25?
1.6.9 If an object’s acceleration is negative, at that particular instant the object can be

(a) slowing down only.
(b) speeding up only.
(c) slowing down or momentarily stopped.
(d) slowing down, momentarily stopped, or speeding up.

1.6.10 In *Star Trek: First Contact*, Worf almost gets knocked into space by the Borg. Assume he was knocked into space and his space suit was equipped with thrusters. Worf fires his thrusters for 1 second, which produces a constant acceleration in the positive direction. In the next second he turns off his thrusters. In the third second he fires his thruster producing a constant negative acceleration. The acceleration as a function of time is given in Figure 2.31. Which of the following graphs represent his position as a function of time?
The position of a moving car is given by the function \( s(t) = 3t^2 + 3 \), where \( t \) is in seconds, and \( s \) is in feet. What function gives the car’s acceleration?

(a) \( a(t) = 3 \)
(b) \( a(t) = 6t \)
(c) \( a(t) = 6 \)
(d) \( a(t) = 6t + 3 \)
(e) \( a(t) = 9 \)
1.7 Limits, Continuity, and Differentiability

Preview Activity 1.8. A function \( f \) defined on \(-4 < x < 4\) is given by the graph in Figure 1.12. Use the graph to answer each of the following questions. Note: to the right of \( x = 2 \), the graph of \( f \) is exhibiting infinite oscillatory behavior.

(a) For each of the values \( a = -3, -2, -1, 0, 1, 2, 3 \), determine whether or not \( \lim_{{x \to a}} f(x) \) exists. If the function has a limit \( L \) at a given point, state the value of the limit using the notation \( \lim_{{x \to a}} f(x) = L \). If the function does not have a limit at a given point, write a sentence to explain why.

(b) For each of the values of \( a \) from part (a) where \( f \) has a limit, determine the value of \( f(a) \) at each such point. In addition, for each such \( a \) value, does \( f(a) \) have the same value as \( \lim_{{x \to a}} f(x) \)?

(c) For each of the values \( a = -3, -2, -1, 0, 1, 2, 3 \), determine whether or not \( f'(a) \) exists. In particular, based on the given graph, ask yourself if it is reasonable to say that \( f \) has a tangent line at \((a, f(a))\) for each of the given \( a \)-values. If so, visually estimate the slope of the tangent line to find the value of \( f'(a) \).
Activity 1.18.

Consider a function that is piecewise-defined according to the formula

\[
  f(x) = \begin{cases} 
    3(x + 2) + 2 & \text{for } -3 < x < -2 \\
    \frac{2}{3}(x + 2) + 1 & \text{for } -2 \leq x < -1 \\
    \frac{2}{3}(x + 2) + 1 & \text{for } -1 < x < 1 \\
    2 & \text{for } x = 1 \\
    4 - x & \text{for } x > 1 
  \end{cases}
\]

Use the given formula to answer the following questions.

(a) For each of the values \( a = -2, -1, 0, 1, 2 \), compute \( f(a) \).

(b) For each of the values \( a = -2, -1, 0, 1, 2 \), determine \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \).

(c) For each of the values \( a = -2, -1, 0, 1, 2 \), determine \( \lim_{x \to a} f(x) \). If the limit fails to exist, explain why by discussing the left- and right-hand limits at the relevant \( a \)-value.

(d) For which values of \( a \) is the following statement true?

\[
  \lim_{x \to a} f(x) \neq f(a)
\]

(e) On the axes provided in Figure 1.13, sketch an accurate, labeled graph of \( y = f(x) \). Be sure to carefully use open circles (○) and filled circles (●) to represent key points on the graph, as dictated by the piecewise formula.

---

Figure 1.13: Axes for plotting the function \( y = f(x) \) in Activity 1.18.
Activity 1.19.

This activity builds on your work in Preview Activity 1.8, using the same function $f$ as given by the graph that is repeated in Figure 1.14.

![Graph of $f(x)$](image)

Figure 1.14: The graph of $y = f(x)$ for Activity 1.19.

(a) At which values of $a$ does $\lim_{x \to a} f(x)$ not exist?

(b) At which values of $a$ is $f(a)$ not defined?

(c) At which values of $a$ does $f$ have a limit, but $\lim_{x \to a} f(x) \neq f(a)$?

(d) State all values of $a$ for which $f$ is not continuous at $x = a$.

(e) Which condition is stronger, and hence implies the other: $f$ has a limit at $x = a$ or $f$ is continuous at $x = a$? Explain, and hence complete the following sentence: “If $f$ \underline{________________} at $x = a$, then $f$ \underline{________________} at $x = a$,” where you complete the blanks with has a limit and is continuous, using each phrase once.
Activity 1.20.

In this activity, we explore two different functions and classify the points at which each is not differentiable. Let $g$ be the function given by the rule $g(x) = |x|$, and let $f$ be the function that we have previously explored in Preview Activity 1.8, whose graph is given again in Figure 1.15.

(a) Reasoning visually, explain why $g$ is differentiable at every point $x$ such that $x \neq 0$.

(b) Use the limit definition of the derivative to show that $g'(0) = \lim_{h \to 0} \frac{|h|}{h}$.

(c) Explain why $g'(0)$ fails to exist by using small positive and negative values of $h$.

(d) State all values of $a$ for which $f$ is not differentiable at $x = a$. For each, provide a reason for your conclusion.

(e) True or false: if a function $p$ is differentiable at $x = b$, then $\lim_{x \to b} p(x)$ must exist. Why?

Figure 1.15: The graph of $y = f(x)$ for Activity 1.20.
1.7. LIMITS, CONTINUITY, AND DIFFERENTIABILITY

Voting Questions

1.7.1 A drippy faucet adds one milliliter to the volume of water in a tub at precisely one-second intervals. Let $f$ be the function that represents the volume of water in the tub at time $t$. Which of the following statements is correct?

(a) $f$ is a continuous function at every time $t$
(b) $f$ is continuous for all $t$ other than the precise instants when the water drips into the tub.
(c) $f$ is not continuous at any time $t$.
(d) There is not enough information to know where $f$ is continuous.

1.7.2 A drippy faucet adds one milliliter to the volume of water in a tub at precisely one-second intervals. Let $g$ be the function that represents the volume of water in the tub as a function of the depth of the water, $x$, in the tub. Which of the following statements is correct?

(a) $g$ is a continuous function at every depth $x$.
(b) there are some values of $x$ at which $g$ is not continuous.
(c) $g$ is not continuous at any depth, $x$.
(d) not enough information is given to know where $g$ is continuous.

1.7.3 You know the following statement is true:

If $f(x)$ is a polynomial, then $f(x)$ is continuous.

Which of the following is also true?

(a) If $f(x)$ is not continuous, then it is not a polynomial.
(b) If $f(x)$ is continuous, then it is a polynomial.
(c) If $f(x)$ is not a polynomial, then it is not continuous.

1.7.4 True or False: You were once exactly 3 feet tall.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
1.7.5 True or False: At some time since you were born your weight in pounds equaled your height in inches.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.7.6 True or False: Along the Equator, there are two diametrically opposite sites that have exactly the same temperature at the same time.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.7.7 Suppose that during half-time at a basketball game the score of the home team was 36 points. True or False: There had to be at least one moment in the first half when the home team had exactly 25 points.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.7.8 At what point on the interval \([-7, 2]\) does the function \(f(x) = \frac{3x^2}{4e^x - 4}\) have a discontinuity?

(a) \(x = 0\)
(b) \(x = 1\)
(c) \(x = 3\)
(d) \(x = 4\)
(e) There is no discontinuity on this interval.

1.7.9 For what value of the constant \(c\) is the function \(f(x)\) continuous, if

\[
f(x) = \begin{cases} 
  cx + 9 & \text{if } x \in (-\infty, 5) \\
  cx^2 - 9 & \text{if } x \in (5, \infty)
\end{cases}
\]
1.7. LIMITS, CONTINUITY, AND DIFFERENTIABILITY

(a) \( c = -\frac{9}{5} \)
(b) \( c = \frac{9}{10} \)
(c) \( c = \frac{9}{25} \)
(d) This is not possible.

1.7.10 Your mother says “If you eat your dinner, you can have dessert.” You know this means, “If you don’t eat your dinner, you cannot have dessert.” Your calculus teacher says, “If \( f \) is differentiable at \( x \), \( f \) is continuous at \( x \).” You know this means

(a) if \( f \) is not continuous at \( x \), \( f \) is not differentiable at \( x \).
(b) if \( f \) is not differentiable at \( x \), \( f \) is not continuous at \( x \).
(c) knowing \( f \) is not continuous at \( x \), does not give us enough information to deduce anything about whether the derivative of \( f \) exists at \( x \).

1.7.11 If \( f'(a) \) exists, \( \lim_{x \to a} f(x) \)

(a) must exist, but there is not enough information to determine it exactly.
(b) equals \( f(a) \).
(c) equals \( f'(a) \).
(d) may not exist.

1.7.12 **True or False:** The function \( f(x) = x^{1/3} \) is continuous at \( x = 0 \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

1.7.13 **True or False:** If \( f(x) = x^{1/3} \) then there is a tangent line at (0,0).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
1.7.14 **True or False:** If \( f(x) = x^{1/3} \) then \( f'(0) \) exists.

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident
1.8 The Tangent Line Approximation

Preview Activity 1.9. Consider the function \( y = g(x) = -x^2 + 3x + 2 \).

(a) Use the limit definition of the derivative to compute a formula for \( y = g'(x) \).

(b) Determine the slope of the tangent line to \( y = g(x) \) at the value \( x = 2 \).

(c) Compute \( g(2) \).

(d) Find an equation for the tangent line to \( y = g(x) \) at the point \((2, g(2))\). Write your result in point-slope form\(^1\).

(e) On the axes provided in Figure 1.16, sketch an accurate, labeled graph of \( y = g(x) \) along with its tangent line at the point \((2, g(2))\).

\(^1\)Recall that a line with slope \( m \) that passes through \((x_0, y_0)\) has equation \( y - y_0 = m(x - x_0) \), and this is the point-slope form of the equation.
Activity 1.21.

Suppose it is known that for a given differentiable function \( y = g(x) \), its local linearization at the point where \( a = -1 \) is given by \( L(x) = -2 + 3(x + 1) \).

(a) Compute the values of \( L(-1) \) and \( L'(-1) \).

(b) What must be the values of \( g(-1) \) and \( g'(-1) \)? Why?

(c) Do you expect the value of \( g(-1.03) \) to be greater than or less than the value of \( g(-1) \)? Why?

(d) Use the local linearization to estimate the value of \( g(-1.03) \).

(e) Suppose that you also know that \( g''(-1) = 2 \). What does this tell you about the graph of \( y = g(x) \) at \( a = -1 \)?

(f) For \( x \) near \(-1\), sketch the graph of the local linearization \( y = L(x) \) as well as a possible graph of \( y = g(x) \) on the axes provided in Figure 1.17.

Figure 1.17: Axes for plotting \( y = L(x) \) and \( y = g(x) \).
Activity 1.22.

The circumference of a sphere was measured to be 71.0 cm with a possible error of 0.5 cm. In this activity we’ll use linear approximation to estimate the maximum error in the calculated surface area.

(a) Write the formula for the surface area of a sphere in terms of the radius, and write the formula for the circumference of a circle in terms of the radius.

(b) Fill in the blanks with the help of equation ??.

\[ \Delta C = \text{__________} \Delta r \]
\[ \Delta S = \text{__________} \Delta r \]

(c) Use your answer from part (b) along with the fact that \( \Delta C = 0.5 \) and \( C = 71 \) to calculate the error in surface area: \( \Delta S \).

(d) Estimate the relative error (fractional error) in the calculated surface area.
Activity 1.23.

Use linear approximation to approximate $\sqrt{4.1}$ using the following hints:

- Let $f(x) = \sqrt{x}$ and find the equation of the tangent line to $f(x)$ at a “nice” point near 4.1.
- Then use this to approximate $\sqrt{4.1}$. 
Activity 1.24.

This activity concerns a function \( f(x) \) about which the following information is known:

- \( f \) is a differentiable function defined at every real number \( x \)
- \( f(2) = -1 \)
- \( y = f'(x) \) has its graph given in Figure 1.18

Figure 1.18: At center, a graph of \( y = f'(x) \); at left, axes for plotting \( y = f(x) \); at right, axes for plotting \( y = f''(x) \).

Your task is to determine as much information as possible about \( f \) (especially near the value \( a = 2 \)) by responding to the questions below.

(a) Find a formula for the tangent line approximation, \( L(x) \), to \( f \) at the point \((2, -1)\).

(b) Use the tangent line approximation to estimate the value of \( f(2.07) \). Show your work carefully and clearly.

(c) Sketch a graph of \( y = f''(x) \) on the righthand grid in Figure 1.18; label it appropriately.

(d) Is the slope of the tangent line to \( y = f(x) \) increasing, decreasing, or neither when \( x = 2 \)? Explain.

(e) Sketch a possible graph of \( y = f(x) \) near \( x = 2 \) on the lefthand grid in Figure 1.18. Include a sketch of \( y = L(x) \) (found in part (a)). Explain how you know the graph of \( y = f(x) \) looks like you have drawn it.

(f) Does your estimate in (b) over- or under-estimate the true value of \( f(2) \)? Why?
1.8. THE TANGENT LINE APPROXIMATION

Voting Questions

1.8.1 If $e^{0.5}$ is approximated by using the tangent line to the graph of $f(x) = e^x$ at $(0,1)$, and we know $f'(0) = 1$, the approximation is

(a) $0.5$
(b) $1 + e^{0.5}$
(c) $1 + 0.5$

1.8.2 The line tangent to the graph of $f(x) = \sin x$ at $(0,0)$ is $y = x$. This implies that

(a) $\sin(0.0005) \approx 0.0005$
(b) The line $y = x$ touches the graph of $f(x) = \sin x$ at exactly one point, $(0,0)$.
(c) $y = x$ is the best straight line approximation to the graph of $f$ for all $x$.

1.8.3 The line $y = 1$ is tangent to the graph of $f(x) = \cos x$ at $(0,1)$. This means that

(a) the only $x$-values for which $y = 1$ is a good estimate for $y = \cos x$ are those that are close enough to 0.
(b) tangent lines can intersect the graph of $f$ infinitely many times.
(c) the farther $x$ is from 0, the worse the linear approximation is.
(d) All of the above

1.8.4 Suppose that $f''(x) < 0$ for $x$ near a point $a$. Then the linearization of $f$ at $a$ is

(a) an over approximation
(b) an under approximation
(c) unknown without more information.

1.8.5 Peeling an orange changes its volume $V$. What does $\Delta V$ represent?

(a) the volume of the rind
(b) the surface area of the orange
(c) the volume of the “edible part” of the orange
(d) $-1 \times$ (the volume of the rind)
1.8.6 You wish to approximate $\sqrt{9.3}$. You know the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ where $x = 9$. What value do you put into the tangent line equation to approximate $\sqrt{9.3}$?

(a) $\sqrt{9.3}$
(b) 9
(c) 9.3
(d) 0.3

1.8.7 We can use a tangent line approximation to $\sqrt{x}$ to approximate square roots of numbers. If we do that for each of the square roots below, for which one would we get the smallest error?

(a) $\sqrt{4.2}$
(b) $\sqrt{4.5}$
(c) $\sqrt{9.2}$
(d) $\sqrt{9.5}$
(e) $\sqrt{16.2}$
(f) $\sqrt{16.5}$
Chapter 2

Computing Derivatives

2.1 Elementary derivative rules

Preview Activity 2.1. Functions of the form \( f(x) = x^n \), where \( n = 1, 2, 3, \ldots \), are often called power functions.

(a) Use the limit definition of the derivative to find \( f'(x) \) for \( f(x) = x^2 \).

(b) Use the limit definition of the derivative to find \( f'(x) \) for \( f(x) = x^3 \).

(c) Use the limit definition of the derivative to find \( f'(x) \) for \( f(x) = x^4 \). (Hint: \( (a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \). Apply this rule to \( (x + h)^4 \) within the limit definition.)

(d) Based on your work in (a), (b), and (c), what do you conjecture is the derivative of \( f(x) = x^5 \)? Of \( f(x) = x^{13} \)?

(e) Conjecture a formula for the derivative of \( f(x) = x^n \) that holds for any positive integer \( n \). That is, given \( f(x) = x^n \) where \( n \) is a positive integer, what do you think is the formula for \( f'(x) \)?
Activity 2.1.

Use the three rules above to determine the derivative of each of the following functions. For each, state your answer using full and proper notation, labeling the derivative with its name. For example, if you are given a function $h(z)$, you should write 

$$h'(z) = \text{ or } \frac{dh}{dz} =$$

as part of your response.

(a) $f(t) = \pi$

(b) $g(z) = 7^z$

(c) $h(w) = w^{3/4}$

(d) $p(x) = 3^{1/2}$

(e) $r(t) = (\sqrt{2})^t$

(f) $\frac{d}{dq}[q^{-1}]$

(g) $m(t) = \frac{1}{t^3}$
Activity 2.2.

Use only the rules for constant, power, and exponential functions, together with the Constant Multiple and Sum Rules, to compute the derivative of each function below with respect to the given independent variable. Note well that we do not yet know any rules for how to differentiate the product or quotient of functions. This means that you may have to do some algebra first on the functions below before you can actually use existing rules to compute the desired derivative formula. In each case, label the derivative you calculate with its name using proper notation such as $f'(x)$, $h'(z)$, $dr/dt$, etc.

(a) \( f(x) = x^{5/3} - x^4 + 2^x \)

(b) \( g(x) = 14e^x + 3x^5 - x \)

(c) \( h(z) = \sqrt{z} + \frac{1}{z} + 5^z \)

(d) \( r(t) = \sqrt{53}t^7 - \pi e^t + e^4 \)

(e) \( s(y) = (y^2 + 1)(y^2 - 1) \)

(f) \( q(x) = \frac{x^3 - x + 2}{x} \)

(g) \( p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12 \)
Activity 2.3.

Each of the following questions asks you to use derivatives to answer key questions about functions. Be sure to think carefully about each question and to use proper notation in your responses.

(a) Find the slope of the tangent line to \( h(z) = \sqrt{z} + \frac{1}{z} \) at the point where \( z = 4 \).

(b) A population of cells is growing in such a way that its total number in millions is given by the function \( P(t) = 2(1.37)^t + 32 \), where \( t \) is measured in days.

   i. Determine the instantaneous rate at which the population is growing on day 4, and include units on your answer.

   ii. Is the population growing at an increasing rate or growing at a decreasing rate on day 4? Explain.

(c) Find an equation for the tangent line to the curve \( p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12 \) at the point where \( a = -1 \).

(d) What the difference between being asked to find the slope of the tangent line (asked in (a)) and the equation of the tangent line (asked in (c))? 

 nhãn
Voting Questions

2.1.1 If $f(x) = 2x^2$, then what is $f'(x)$?

(a) $2x$
(b) $2x^2$
(c) 4
(d) $4x$
(e) $4x^2$
(f) Cannot be determined from what we know

2.1.2 If $f(x) = 7$, then what is $f'(x)$?

(a) 7
(b) $7x$
(c) 0
(d) 1
(e) Cannot be determined from what we know

2.1.3 If $f(x) = 2x^{2.5}$, then what is $f'(x)$?

(a) $2.5x^{2.5}$
(b) $5x^{2.5}$
(c) $2.5x^{1.5}$
(d) $5x^{1.5}$
(e) Cannot be determined from what we know

2.1.4 If $f(x) = \pi^2$, then what is $f'(x)$?

(a) $2\pi$
(b) $\pi^2$
(c) 0
(d) 2
(e) Cannot be determined from what we know
2.1.5 If \( f(x) = 3^x \), then what is \( f'(x) \)?

(a) \( x \cdot 3^{x-1} \)
(b) \( 3^x \)
(c) \( 3x^2 \)
(d) 0
(e) Cannot be determined from what we know

2.1.6 If \( f(x) = 4\sqrt{x} \), then what is \( f'(x) \)?

(a) \( 4\sqrt{x} \)
(b) \( 2\sqrt{x} \)
(c) \( 2x^{1/2} \)
(d) \( 4x^{-1/2} \)
(e) \( 2x^{-1/2} \)
(f) Cannot be determined from what we know

2.1.7 If \( f(t) = 3t^2 + 2t \), then what is \( f'(t) \)?

(a) \( 3t^2 + 2 \)
(b) \( 6t + 2 \)
(c) \( 9t^2 + 2t \)
(d) \( 9t + 2 \)
(e) Cannot be determined from what we know

2.1.8 Let \( f(x) = -16x^2 + 96x \). Find \( f'(2) \).

(a) 0
(b) 32
(c) 128
(d) \( f'(2) \) does not exist.

2.1.9 If \( a + b^2 = 3 \), find \( \frac{da}{db} \).

(a) \( \frac{da}{db} = 0 \)
2.1. ELEMENTARY DERIVATIVE RULES

(b) \( \frac{da}{db} = 2b \)
(c) \( \frac{da}{db} = -2b \)
(d) Cannot be determined from this expression

2.1.10 If \( r(q) = 4q^{-5} \), then what is \( r'(q) \)?

(a) \( 5q^{-5} \)
(b) \( -20q^{-4} \)
(c) \( -20q^{-5} \)
(d) \( -20q^{-6} \)
(e) Cannot be determined from what we know

2.1.11 If \( f(x) = x(x + 5) \), then what is \( f'(x) \)?

(a) \( x + 5 \)
(b) \( 1 \)
(c) \( 2x + 5 \)
(d) \( 2x \)
(e) Cannot be determined from what we know

2.1.12 If \( f(x) = \frac{2}{x^3} \), then what is \( f'(x) \)?

(a) \( \frac{2}{3x^4} \)
(b) \( \frac{-6}{x^4} \)
(c) \( 6x^{-2} \)
(d) \( -3x^{-4} \)
(e) Cannot be determined from what we know

2.1.13 If \( f(x) = x^2 + \frac{3}{x} \), then what is \( f'(x) \)?

(a) \( 2x - 3x^{-2} \)
(b) \( 2x + 3x^{-1} \)
(c) \( 2x - 3x^2 \)
(d) \( x^2 - 3x^{-1} \)
(e) Cannot be determined from what we know
2.1.14 If \( f(x) = 4\sqrt{x} + \frac{5}{x^2} \), then what is \( f'(x) \)?

(a) \( 2x^{-1/2} - 10x^{-3} \)
(b) \( 4x^{1/2} + 5x^{-2} \)
(c) \( 2x^{1/2} - 10x^{-3} \)
(d) \( 2x^{-1/2} + 10x^{-3} \)
(e) Cannot be determined from what we know

2.1.15 If \( f(x) = \frac{x^2 + 5x}{x}, \) then what is \( f'(x) \)?

(a) \( 2x + 5 \)
(b) \( x + 5 \)
(c) 1
(d) 0
(e) Cannot be determined from what we know

2.1.16 If \( f(x) = \frac{x}{x^2 + 5x} \), then what is \( f'(x) \)?

(a) \( \frac{1}{2x+5} \)
(b) \( -x^{-2} \)
(c) \( \frac{1}{x} + \frac{1}{5} \)
(d) 1
(e) Cannot be determined from what we know

2.1.17 If \( f(m) = am^2 + bm \), then what is \( f'(m) \)?

(a) \( m^2 + m \)
(b) \( 2am + b \)
(c) \( am \)
(d) 0
(e) Cannot be determined from what we know

2.1.18 If \( p(q) = \frac{2q-8}{q^2} \), then what is \( p'(2) \)?

(a) \( \frac{2}{2q} \)
2.1. ELEMENTARY DERIVATIVE RULES

(b) \(-2q^{-2} + 16q^{-3}\)
(c) \(\frac{1}{2}\)
(d) \(\frac{3}{2}\)
(e) 0
(f) Cannot be determined from what we know

2.1.19 If \(f(d) = ad^2 + bd + d + c\), then what is \(f'(d)\)?

(a) \(2ad + b + d\)
(b) \(2ad + b + 1\)
(c) \(2ad + b + c\)
(d) \(2ad + b\)
(e) \(2ad + b + 1 + c\)
(f) \(2ad + b + 2\)

2.1.20 If \(g(d) = ab^2 + 3c^3d + 5b^2c^2d^2\), then what is \(g''(d)\)?

(a) \(3c^3 + 10b^2c^2d\)
(b) \(10b^2c^2\)
(c) \(42 + 18cd\)
(d) \(2ab + 9c^2d + 40bcd\)
(e) Cannot be determined from what we know

2.1.21 Find the equation of the line that is tangent to the function \(f(x) = 3x^2\) when \(x = 2\). Recall that this line not only has the same slope as \(f(x)\) at \(x = 2\), but also has the same value of \(y\) when \(x = 2\).

(a) \(y = 12x - 12\)
(b) \(y = 6x\)
(c) \(y = 3x + 6\)
(d) \(y = 12x\)
(e) \(y = 6x + 6\)

2.1.22 Which is the equation of the line tangent to \(y = x^2\) at \(x = 4\)?
2.1. ELEMENTARY DERIVATIVE RULES

(a) \( y = (2x)x + 4 \)
(b) \( y = 8x + 4 \)
(c) \( y = 8x - 16 \)
(d) \( y = 16x - 48 \)

2.1.23 A ball is thrown into the air and its height \( h \) (in meters) after \( t \) seconds is given by the function \( h(t) = 10 + 20t - 5t^2 \). When the ball reaches its maximum height, its velocity will be zero. At what time will the ball reach its maximum height?

(a) \( t = 0 \) seconds
(b) \( t = 1 \) second
(c) \( t = 2 \) seconds
(d) \( t = 3 \) seconds
(e) \( t = 4 \) seconds

2.1.24 A ball is thrown into the air and its height \( h \) (in meters) after \( t \) seconds is given by the function \( h(t) = 10 + 20t - 5t^2 \). When the ball reaches its maximum height, its velocity will be zero. What will be the ball’s maximum height?

(a) \( h = 10 \) meters
(b) \( h = 20 \) meters
(c) \( h = 30 \) meters
(d) \( h = 40 \) meters
(e) \( h = 50 \) meters

2.1.25 Suppose a stone is thrown vertically upward with an initial velocity of 64 ft/s from a bridge 96 ft above a river. By Newton’s laws of motion, the position of the stone (measured as the height above the ground) after \( t \) seconds is \( s(t) = -16t^2 + 64t + 96 \). How many seconds after it is thrown will the stone reach its maximum height?

(a) \( (2 - \sqrt{10}) \) s
(b) \( 2 \) s
(c) \( (2 + \sqrt{10}) \) s
(d) \( 4 \) s

2.1.26 \( \frac{d}{dx} (e^x) \) is
2.1. ELEMENTARY DERIVATIVE RULES

(a) $xe^{x-1}$
(b) $e^x$
(c) $e^x \ln x$
(d) 0
(e) Cannot be determined from what we know

2.1.27 $\frac{d}{dx} (5^x)$ is
(a) $x5^{x-1}$
(b) $5^x$
(c) $5^x \ln x$
(d) $5^x \ln 5$
(e) Cannot be determined from what we know

2.1.28 $\frac{d}{dx} (x^e)$ is
(a) $ex^{e-1}$
(b) $x^e$
(c) $x^e \ln x$
(d) $ex$
(e) Cannot be determined from what we know

2.1.29 $\frac{d}{dx} (e^7)$ is
(a) $7e^6$
(b) $e^7$
(c) $e^7 \ln 7$
(d) 0
(e) Cannot be determined from what we know

2.1.30 $\frac{d}{dx} (3e^x)$ is
(a) $3xe^{x-1}$
(b) $3e^x$
(c) $e^x \ln 3$
(d) 3
(e) Cannot be determined from what we know

2.1.31 \( \frac{d}{dx} (2 \cdot 5^x) \) is

(a) \( 10^x \)
(b) \( 2 \cdot 5^x \)
(c) \( 10^x \ln 10 \)
(d) \( 2 \cdot 5^x \ln 5 \)
(e) \( 10^x \ln 5 \)
(f) Cannot be determined from what we know

2.1.32 \( \frac{d}{dx} (xe^x) \) is

(a) \( x^2 e^{x-1} \)
(b) \( xe^x \)
(c) \( e^x \ln x \)
(d) Cannot be determined from what we know

2.1.33 If \( \ln x - y = 0 \), find \( \frac{dx}{dy} \).

(a) \( \frac{dx}{dy} = e^x \)
(b) \( \frac{dx}{dy} = e^{-x} \)
(c) \( \frac{dx}{dy} = e^y \)
(d) \( \frac{dx}{dy} = e^{-y} \)
(e) Cannot be determined from this expression

2.1.34 \( \frac{d}{dx} (e^{x+2}) \) is

(a) \( (x + 2)e^{x+1} \)
(b) \( e^x \)
(c) \( e^2 \)
(d) 0
(e) Cannot be determined from what we know
2.1. ELEMENTARY DERIVATIVE RULES

2.1.35 $\frac{d}{dx}(e^{2x})$ is

(a) $e^{2x}$
(b) $e^{2}e^{x}$
(c) 0
(d) Cannot be determined from what we know

2.1.36 If $u = 5^v$, find $\frac{d^2u}{dv^2}$.

(a) $\frac{d^2u}{dv^2} = 0$
(b) $\frac{d^2u}{dv^2} = 5^v$
(c) $\frac{d^2u}{dv^2} = 5^v \ln 5$
(d) $\frac{d^2u}{dv^2} = 5^v (\ln 5)^2$
(e) $\frac{d^2u}{dv^2} = v(v - 1)5^{v-2}$
(f) Cannot be determined from what we know

2.1.37 If $u = ve^w + xy^v$, find $\frac{du}{dv}$.

(a) $\frac{du}{dv} = e^w + x y^v \ln y$
(b) $\frac{du}{dv} = ve^w + x y^v \ln y$
(c) $\frac{du}{dv} = e^w + x y^v \ln v$
(d) $\frac{du}{dv} = ve^w + x y^v \ln v$
(e) Cannot be determined from what we know

2.1.38 Find the equation of the line that is tangent to the function $g(x) = 2e^x$ at $x = 1$.

(a) $y = 2e^x x$
(b) $y = 2ex$
(c) $y = 2e^x x + 2e$
(d) $y = 2ex + 2e$
(e) None of the above
2.2 The sine and cosine functions

Preview Activity 2.2. Consider the function \( g(x) = 2^x \), which is graphed in Figure 2.1.

(a) At each of \( x = -2, -1, 0, 1, 2 \), use a straightedge to sketch an accurate tangent line to \( y = g(x) \).

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point in (a).

(c) Use the limit definition of the derivative to estimate \( g'(0) \) by using small values of \( h \), and compare the result to your visual estimate for the slope of the tangent line to \( y = g(x) \) at \( x = 0 \) in (b).

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of \( y = g'(x) \) on the axes adjacent to the graph of \( y = g(x) \).

(e) Write at least one sentence that explains why it is reasonable to think that \( g'(x) = cg(x) \), where \( c \) is a constant. In addition, calculate \( \ln(2) \), and then discuss how this value, combined with your work above, reasonably suggests that \( g'(x) = 2^x \ln(2) \).

Figure 2.1: At left, the graph of \( y = g(x) = 2^x \). At right, axes for plotting \( y = g'(x) \).
Activity 2.4.
Consider the function \( f(x) = \sin(x) \), which is graphed in Figure 2.2 below. Note carefully that the grid in the diagram does not have boxes that are \( 1 \times 1 \), but rather approximately \( 1.57 \times 1 \), as the horizontal scale of the grid is \( \pi/2 \) units per box.

(a) At each of \( x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \), use a straightedge to sketch an accurate tangent line to \( y = f(x) \).

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Pay careful attention to the scale of the grid.

(c) Use the limit definition of the derivative to estimate \( f'(0) \) by using small values of \( h \), and compare the result to your visual estimate for the slope of the tangent line to \( y = f(x) \) at \( x = 0 \) in (b). Using periodicity, what does this result suggest about \( f'(2\pi) \)? about \( f'(-2\pi) \)?

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of \( y = f'(x) \) on the axes adjacent to the graph of \( y = f(x) \).

(e) What familiar function do you think is the derivative of \( f(x) = \sin(x) \)?

---

Figure 2.2: At left, the graph of \( y = f(x) = \sin(x) \).
Activity 2.5.

Consider the function \( g(x) = \cos(x) \), which is graphed in Figure 2.3 below. Note carefully that the grid in the diagram does not have boxes that are \( 1 \times 1 \), but rather approximately \( 1.57 \times 1 \), as the horizontal scale of the grid is \( \pi/2 \) units per box.

![Figure 2.3](image)

**Figure 2.3:** At left, the graph of \( y = g(x) = \cos(x) \).

(a) At each of \( x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \), use a straightedge to sketch an accurate tangent line to \( y = g(x) \).

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Again, note the scale of the axes and grid.

(c) Use the limit definition of the derivative to estimate \( g'(\frac{\pi}{2}) \) by using small values of \( h \), and compare the result to your visual estimate for the slope of the tangent line to \( y = g(x) \) at \( x = \frac{\pi}{2} \) in (b). Using periodicity, what does this result suggest about \( g'(-\frac{3\pi}{2}) \)? Can symmetry on the graph help you estimate other slopes easily?

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of \( y = g'(x) \) on the axes adjacent to the graph of \( y = g(x) \).

(e) What familiar function do you think is the derivative of \( g(x) = \cos(x) \)?

\(<\)
Activity 2.6.

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

(a) Determine the derivative of \( h(t) = 3 \cos(t) - 4 \sin(t) \).

(b) Find the exact slope of the tangent line to \( y = f(x) = 2x + \frac{\sin(x)}{2} \) at the point where \( x = \frac{\pi}{6} \).

(c) Find the equation of the tangent line to \( y = g(x) = x^2 + 2 \cos(x) \) at the point where \( x = \frac{\pi}{2} \).

(d) Determine the derivative of \( p(z) = z^4 + 4z^2 + 4 \cos(z) - \sin\left(\frac{\pi}{2}\right) \).

(e) The function \( P(t) = 24 + 8 \sin(t) \) represents a population of a particular kind of animal that lives on a small island, where \( P \) is measured in hundreds and \( t \) is measured in decades since January 1, 2010. What is the instantaneous rate of change of \( P \) on January 1, 2030? What are the units of this quantity? Write a sentence in everyday language that explains how the population is behaving at this point in time.
Voting Questions

2.2.1 \( \frac{d}{dx} (-3 \sin x) \) is

(a) \( \cos x \)
(b) \( -3 \sin x \)
(c) \( 3 \cos x \)
(d) \( -3 \cos x \)

2.2.2 \( \frac{d}{dx} \frac{\cos x}{2x} \) is

(a) \( (\sin x)/25 \)
(b) \( -\sin x \)
(c) \( (-\sin x)/25 \)
(d) \( (-\cos x)/25 \)

2.2.3 The 4th derivative of \( \sin x \) is

(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
(d) \( -\cos x \)

2.2.4 The 10th derivative of \( \sin x \) is

(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
(d) \( -\cos x \)

2.2.5 The 100th derivative of \( \sin x \) is

(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
2.2.6 The 41st derivative of \( \sin x \) is

(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
(d) \( -\cos x \)

2.2.7 The equation of the line tangent to the graph of \( \cos x \) at \( x = 0 \) is

(a) \( y = 1 \)
(b) \( y = 0 \)
(c) \( y = \cos x \)
(d) \( y = x \)

2.2.8 The 30th derivative of \( \cos x \) is

(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
(d) \( -\cos x \)
2.3 The product and quotient rules

Preview Activity 2.3. Let \( u \) and \( v \) be the functions defined by \( u(t) = 2t^2 \) and \( v(t) = t^3 + 4t \).

(a) Determine \( u'(t) \) and \( v'(t) \).

(b) Let \( p(t) = 2t^2(t^3 + 4t) \) and observe that \( p(t) = u(t) \cdot v(t) \). Rewrite the formula for \( p \) by distributing the \( 2t^2 \) term. Then, compute \( p'(t) \) using the sum and constant multiple rules.

(c) True or false: \( p'(t) = u'(t) \cdot v'(t) \).

(d) Let \( q(t) = \frac{t^3 + 4t}{2t^2} \) and observe that \( q(t) = \frac{v(t)}{u(t)} \). Rewrite the formula for \( q \) by dividing each term in the numerator by the denominator and simplify to write \( q \) as a sum of constant multiples of powers of \( t \). Then, compute \( q'(t) \) using the sum and constant multiple rules.

(e) True or false: \( q'(t) = \frac{v'(t)}{u'(t)} \).
Activity 2.7.

Use the product rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. That is, if you’re given a formula for \( f(x) \), clearly label the formula you find for \( f'(x) \). It is not necessary to algebraically simplify any of the derivatives you compute.

(a) Let \( m(w) = 3w^{17}4^w \). Find \( m'(w) \).

(b) Let \( h(t) = (\sin(t) + \cos(t))t^4 \). Find \( h'(t) \).

(c) Determine the slope of the tangent line to the curve \( y = f(x) \) at the point where \( a = 1 \) if \( f \) is given by the rule \( f(x) = e^x \sin(x) \).

(d) Find the tangent line approximation \( L(x) \) to the function \( y = g(x) \) at the point where \( a = -1 \) if \( g \) is given by the rule \( g(x) = (x^2 + x)^{2x} \).
Activity 2.8.

Use the quotient rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. That is, if you’re given a formula for \( f(x) \), clearly label the formula you find for \( f'(x) \). It is not necessary to algebraically simplify any of the derivatives you compute.

(a) Let \( r(z) = \frac{3z}{z^4 + 1} \). Find \( r'(z) \).

(b) Let \( v(t) = \frac{\sin(t)}{\cos(t) + t^2} \). Find \( v'(t) \).

(c) Determine the slope of the tangent line to the curve \( R(x) = \frac{x^2 - 2x - 8}{x^2 - 9} \) at the point where \( x = 0 \).

(d) When a camera flashes, the intensity \( I \) of light seen by the eye is given by the function

\[
I(t) = \frac{100t}{e^t},
\]

where \( I \) is measured in candles and \( t \) is measured in milliseconds. Compute \( I'(0.5) \), \( I'(2) \), and \( I'(5) \); include appropriate units on each value; and discuss the meaning of each.
Activity 2.9.

Use relevant derivative rules to answer each of the questions below. Throughout, be sure to use proper notation and carefully label any derivative you find by name.

(a) Let \( f(r) = (5r^3 + \sin(r))(4r - 2\cos(r)) \). Find \( f'(r) \).

(b) Let \( p(t) = \frac{\cos(t)}{t^6} \cdot 6^t \). Find \( p'(t) \).

(c) Let \( g(z) = 3z^7 e^z - 2z^2 \sin(z) + \frac{z}{z^2 + 1} \). Find \( g'(z) \).

(d) A moving particle has its position in feet at time \( t \) in seconds given by the function \( s(t) = \frac{3\cos(t) - \sin(t)}{e^t} \). Find the particle’s instantaneous velocity at the moment \( t = 1 \).

(e) Suppose that \( f(x) \) and \( g(x) \) are differentiable functions and it is known that \( f(3) = -2 \), \( f'(3) = 7 \), \( g(3) = 4 \), and \( g'(3) = -1 \). If \( p(x) = f(x) \cdot g(x) \) and \( q(x) = \frac{f(x)}{g(x)} \), calculate \( p'(3) \) and \( q'(3) \).
Voting Questions

2.3.1 \( \frac{d}{dx}(x^2e^x) = \)
(a) \( 2xe^x \)
(b) \( x^2e^x \)
(c) \( 2xe^x + x^2e^{x-1} \)
(d) \( 2xe^x + x^2e^x \)

2.3.2 \( \frac{d}{dx}(xe^x) = \)
(a) \( xe^x + x^2e^x \)
(b) \( e^x + xe^x \)
(c) \( 2xe^x + xe^x \)
(d) \( e^x \)

2.3.3 \( \frac{d}{dt}((t^2 + 3)e^t) = \)
(a) \( 2te^t + (t^2 + 3)e^t \)
(b) \( (2t + 3)e^t + (t^2 + 3)e^t \)
(c) \( 2te^t \)
(d) \( 2te^t + t^2e^t \)
(e) \( (t^2 + 3)e^t \)

2.3.4 \( \frac{d}{dx}(x^34^x) = \)
(a) \( 3x^24^x \ln 4 \)
(b) \( x^34^x + x^34^x \ln 4 \)
(c) \( 3x^24^x + x^34^x \)
(d) \( 3x^24^x + x^34^x \ln 4 \)

2.3.5 When differentiating a constant multiple of a function (like \( 3x^2 \)) the Constant Multiple Rule tells us \( \frac{d}{dx}cf(x) = c \frac{d}{dx}f(x) \) and the Product Rule says \( \frac{d}{dx}cf(x) = c \frac{d}{dx}f(x) + f(x) \frac{d}{dx}c \). Do these two rules agree?
(a) Yes, they agree.
(b) No, they do not agree.

2.3.6 \( \frac{d}{dx} xe^x = \)

(a) \( e^x + xe^x \)
(b) \( \frac{e^x - xe^x}{e^{2x}} \)
(c) \( \frac{xe^x - e^x}{e^{2x}} \)
(d) \( \frac{xe^x - e^x}{e^{2x}} \)

2.3.7 \( \frac{d}{dx} \frac{x^{1.5}}{3^x} = \)

(a) \( \frac{1.5x^{0.5} - 3x \ln 3}{3^{2x}} \)
(b) \( \frac{1.5x^{0.5} - 2x^{1.5}x \ln 3}{3^{2x}} \)
(c) \( \frac{1.5x^{0.5} - 3x \ln 3}{1.5x^{0.5}} \)
(d) \( \frac{1.5x^{0.5}3x + x^{1.5}3x \ln 3}{1.5x^{0.5}} \)

2.3.8 If \( e^a - \frac{b}{a^2} = 5 \), find \( \frac{db}{da} \).

(a) \( \frac{db}{da} = e^a \)
(b) \( \frac{db}{da} = a^2 e^a \)
(c) \( \frac{db}{da} = a^2 e^a - 5a^2 \)
(d) \( \frac{db}{da} = 2ae^a + a^2 e^a - 10a \)
(e) \( \frac{db}{da} = 2ae^a + a^2 e^a - 10ae^a - 5a^2 e^a \)
(f) Cannot be determined from this expression

2.3.9 \( \frac{d}{dx} (25x^2 e^x) = \)

(a) \( 50x^2 e^x + 25x^2 e^x \)
(b) \( 25xe^x + 25x^2 e^x \)
(c) \( 50xe^x + 25x^2 e^x \)
(d) \( 50xe^x + 25xe^x \)

2.3.10 \( \frac{d}{dt} \frac{3t+1}{3t+2} = \)
2.3. THE PRODUCT AND QUOTIENT RULES

(a) \( \frac{3(5t+2)-(3t+1)5}{(5t+2)^2} \)

(b) \( \frac{3(5t+2)-(3t+1)5}{(3t+1)^2} \)

(c) \( \frac{(3t+1)(5t+2)-(3t+1)5}{(5t+2)^2} \)

(d) \( \frac{3(5t+2)-(3t+1)(5t+2)}{(5t+2)^2} \)

2.3.11 \( \frac{d}{dt} \sqrt{\frac{t}{t^2+1}} = \)

(a) \( \frac{t^{-1/2} - 2t}{(t^2+1)^2} \)

(b) \( \frac{t^{-1/2} + 2t\sqrt{t}}{(t^2+1)^2} \)

(c) \( \frac{1}{(t^2+1)^2} \)

(d) \( \frac{t^{-1/2}(t^2+1) - 2t\sqrt{t}}{(t^2+1)^2} \)

2.3.12 If \( f(3) = 2, f'(3) = 4, g(3) = 1, g'(3) = 3 \), and \( h(x) = f(x)g(x) \), then what is \( h'(3) \)?

(a) 2
(b) 10
(c) 11
(d) 12
(e) 14

2.3.13 If \( f(3) = 2, f'(3) = 4, g(3) = 1, g'(3) = 3 \), and \( h(x) = \frac{f(x)}{g(x)} \), then what is \( h'(3) \)?

(a) \(-2\)
(b) \(2\)
(c) \(-\frac{2}{9}\)
(d) \(\frac{2}{9}\)
(e) 5

2.3.14 If \( h = \frac{ab^2 + b}{c^3} \) then what is \( \frac{dh}{db} \)?

(a) \( \frac{2aabe^b}{c^3} \)
(b) \( \frac{2aabe^b}{3c^2} \)
(c) \( \frac{2abe + ab^2e^b}{c^3} \)

(d) \( \frac{2abe^b - 3e^{2ab}b^3}{c^6} \)

2.3.15 My old uncle Stanley has a collection of rare and valuable books: He has a total of 4,000 books, that are worth an average of $60 each. His books are rising in value over time, so that each year, the average price per book goes up by $0.50. However he also has to sell 30 books per year in order to pay for his snowboarding activities. The value of the collection is

(a) increasing by approximately $240,000 per year.
(b) increasing by approximately $2000 per year.
(c) increasing by approximately $200 per year.
(d) decreasing by approximately $1,800 per year.
(e) decreasing by approximately $119,970 per year.

2.3.16 The functions \( f(x) \) and \( h(x) \) are plotted below. The function \( g = 2fh \). What is \( g'(2) \)?

(a) \( g'(2) = -1 \)
(b) \( g'(2) = 2 \)
(c) \( g'(2) = 4 \)
(d) \( g'(2) = 32 \)

2.3.17 The 4th derivative of \( \cos x \) is

(a) \( \sin x \)
(b) \( \cos x \)
(c) \( -\sin x \)
(d) \( -\cos x \)
2.3.18 If \( f(x) = \frac{x}{\sin x} \), then \( f'(x) = \)

(a) \( \frac{\sin x - x \cos x}{\sin^2 x} \)
(b) \( \frac{\sin x - x \cos x}{\cos^2 x} \)
(c) \( \frac{x \cos x - x \sin x}{\sin^2 x} \)
(d) \( \frac{\cos x - x \cos x}{\sin^2 x} \)

2.3.19 If \( f(x) = \sin x \cos x \), then \( f'(x) = \)

(a) \( 1 - 2 \sin^2 x \)
(b) \( 2 \cos^2 x - 1 \)
(c) \( \cos 2x \)
(d) All of the above
(e) None of the above

2.3.20 \( \frac{d}{dx} (e^x \sin x) \) is

(a) \( e^x \cos x \)
(b) \( e^x \sin x \)
(c) \( e^x \cos x + e^x \sin x \)
(d) \( e^x \sin x - e^x \cos x \)
2.4 Derivatives of other trigonometric functions

Preview Activity 2.4. Consider the function $f(x) = \tan(x)$, and remember that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

(a) What is the domain of $f$?

(b) Use the quotient rule to show that one expression for $f'(x)$ is

$$f'(x) = \frac{\cos(x) \cos(x) + \sin(x) \sin(x)}{\cos^2(x)}.$$

(c) What is the Fundamental Trigonometric Identity? How can this identity be used to find a simpler form for $f'(x)$?

(d) Recall that $\sec(x) = \frac{1}{\cos(x)}$. How can we express $f'(x)$ in terms of the secant function?

(e) For what values of $x$ is $f'(x)$ defined? How does this set compare to the domain of $f$?
Activity 2.10.

Let \( h(x) = \sec(x) \) and recall that \( \sec(x) = \frac{1}{\cos(x)} \).

(a) What is the domain of \( h \)?

(b) Use the quotient rule to develop a formula for \( h'(x) \) that is expressed completely in terms of \( \sin(x) \) and \( \cos(x) \).

(c) How can you use other relationships among trigonometric functions to write \( h'(x) \) only in terms of \( \tan(x) \) and \( \sec(x) \)?

(d) What is the domain of \( h' \)? How does this compare to the domain of \( h \)?
Activity 2.11.

Let \( p(x) = \csc(x) \) and recall that \( \csc(x) = \frac{1}{\sin(x)} \).

(a) What is the domain of \( p \)?

(b) Use the quotient rule to develop a formula for \( p'(x) \) that is expressed completely in terms of \( \sin(x) \) and \( \cos(x) \).

(c) How can you use other relationships among trigonometric functions to write \( p'(x) \) only in terms of \( \cot(x) \) and \( \csc(x) \)?

(d) What is the domain of \( p' \)? How does this compare to the domain of \( p \)?

\[ \triangle \]
Activity 2.12.

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

(a) Let \( f(x) = 5 \sec(x) - 2 \csc(x) \). Find the slope of the tangent line to \( f \) at the point where \( x = \frac{\pi}{3} \).

(b) Let \( p(z) = z^2 \sec(z) - z \cot(z) \). Find the instantaneous rate of change of \( p \) at the point where \( z = \frac{\pi}{4} \).

(c) Let \( h(t) = \frac{\tan(t)}{t^2 + 1} - 2e^t \cos(t) \). Find \( h'(t) \).

(d) Let \( g(r) = \frac{r sec(r)}{5r} \). Find \( g'(r) \).

(e) When a mass hangs from a spring and is set in motion, the object’s position oscillates in a way that the size of the oscillations decrease. This is usually called a damped oscillation. Suppose that for a particular object, its displacement from equilibrium (where the object sits at rest) is modeled by the function

\[
s(t) = \frac{15 \sin(t)}{e^t}.
\]

Assume that \( s \) is measured in inches and \( t \) in seconds. Sketch a graph of this function for \( t \geq 0 \) to see how it represents the situation described. Then compute \( ds/dt \), state the units on this function, and explain what it tells you about the object’s motion. Finally, compute and interpret \( s'(2) \).
Voting Questions

2.4.1 If \( f(x) = \tan x \), then \( f'(x) = \)

- (a) \( \sec^2 x \)
- (b) \( \cot x \)
- (c) \( -\cot x \)
- (d) All of the above
- (e) None of the above
2.5 The chain rule

Preview Activity 2.5. For each function given below, identify its fundamental algebraic structure. In particular, is the given function a sum, product, quotient, or composition of basic functions? If the function is a composition of basic functions, state a formula for the inner function $u$ and the outer function $f$ so that the overall composite function can be written in the form $f(u(x))$. If the function is a sum, product, or quotient of basic functions, use the appropriate rule to determine its derivative.

(a) $h(x) = \tan(2^x)$
(b) $p(x) = 2^x \tan(x)$
(c) $r(x) = (\tan(x))^2$
(d) $m(x) = e^{\tan(x)}$
(e) $w(x) = \sqrt{x} + \tan(x)$
(f) $z(x) = \sqrt{\tan(x)}$
Activity 2.13.

For each function given below, identify an inner function \( u \) and outer function \( f \) to write the function in the form \( f(u(x)) \). Then, determine \( f'(x) \), \( u'(x) \), and \( f'(u(x)) \), and finally apply the chain rule to determine the derivative of the given function.

(a) \( h(x) = \cos(x^4) \)

(b) \( p(x) = \sqrt{\tan(x)} \)

(c) \( s(x) = 2\sin(x) \)

(d) \( z(x) = \cot^5(x) \)

(e) \( m(x) = (\sec(x) + e^x)^9 \)
Activity 2.14.

For each of the following functions, find the function’s derivative. State the rule(s) you use, label relevant derivatives appropriately, and be sure to clearly identify your overall answer.

(a) \( p(r) = 4\sqrt{r^6} + 2e^r \)

(b) \( m(v) = \sin(v^2) \cos(v^3) \)

(c) \( h(y) = \frac{\cos(10y)}{e^{4y} + 1} \)

(d) \( s(z) = 2z^2 \sec(z) \)

(e) \( c(x) = \sin(e^{x^2}) \)
Activity 2.15.

Use known derivative rules, including the chain rule, as needed to answer each of the following questions.

(a) Find an equation for the tangent line to the curve \( y = \sqrt{e^x + 3} \) at the point where \( x = 0 \).

(b) If \( s(t) = \frac{1}{(t^2 + 1)^3} \) represents the position function of a particle moving horizontally along an axis at time \( t \) (where \( s \) is measured in inches and \( t \) in seconds), find the particle’s instantaneous velocity at \( t = 1 \). Is the particle moving to the left or right at that instant?

(c) At sea level, air pressure is 30 inches of mercury. At an altitude of \( h \) feet above sea level, the air pressure, \( P \), in inches of mercury, is given by the function

\[
P = 30e^{-0.000323h}.
\]

Compute \( dP/dh \) and explain what this derivative function tells you about air pressure, including a discussion of the units on \( dP/dh \). In addition, determine how fast the air pressure is changing for a pilot of a small plane passing through an altitude of 1000 feet.

(d) Suppose that \( f(x) \) and \( u(x) \) are differentiable functions and that the following information about them is known:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( u(x) )</th>
<th>( u'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2</td>
<td>-5</td>
<td>-3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>-3</td>
<td>4</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

If \( C(x) \) is a function given by the formula \( f(u(x)) \), determine \( C'(2) \). In addition, if \( D(x) \) is the function \( f(f(x)) \), find \( D'(-1) \).
Voting Questions

2.5.1 \(\frac{d}{dx} \sin(\cos x)\) is

(a) \(- \cos x \cos(\cos x)\)
(b) \(- \sin x \cos(\sin x)\)
(c) \(- \sin x \sin(\cos x)\)
(d) \(- \sin x \cos(\cos x)\)

2.5.2 We know that \(\frac{d}{dx} \sin x = \cos x\). True or False: \(\frac{d}{dx} \sin(2x) = \cos(2x)\).
(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.5.3 \(\frac{d}{dx} (10 \sin (12x))\) is

(a) 120 \cos(12x)
(b) 10 \cos(12x)
(c) 120 \sin(12x)
(d) \(-120 \cos(12x)\)

2.5.4 \(\frac{d}{dx} (\sin (x^2 + 5))\) is

(a) \cos(x^2 + 5)
(b) \sin(2x + 5)
(c) 2x \sin(x^2 + 5)
(d) 2x \cos(x^2 + 5)

2.5.5 \(\frac{d}{dx} (\sin^2 (ax))\) is

(a) 2 \sin(ax)
(b) 2 \cos(ax)
(c) 2a \sin(ax)
(d) 2a \sin(ax) \cos(ax)
2.5.6  \( \frac{d}{dx} (\sin x + e^{\sin x}) \) is
(a) \( \cos x + e^{\cos x} \)
(b) \( \cos x + e^{\sin x} \)
(c) \( \cos x + e^{\sin x} \cos x \)
(d) None of the above

2.5.7 The equation of the line tangent to the graph of \( 2 \sin 3x \) at \( x = \frac{\pi}{3} \) is
(a) \( y = 6x - 2\pi \)
(b) \( y = 6x \cos 3x - 2\pi \)
(c) \( y = -6x + 2\pi \)
(d) \( y = -6x + 2\pi - 1 \)

2.5.8  \( \frac{d}{dx} (x^2 + 5)^{100} \) =
(a) \( 100(x^2 + 5)^{99} \)
(b) \( 100x(x^2 + 5)^{99} \)
(c) \( 200x(x^2 + 5)^{99} \)
(d) \( 200x(2x + 5)^{99} \)

2.5.9  \( \frac{d}{dx} e^{3x} \) =
(a) \( 3e^{3x} \)
(b) \( e^{3x} \)
(c) \( 3xe^{3x} \)
(d) \( 3e^3 \)

2.5.10  \( \frac{d}{dx} \sqrt{1-x} \) =
(a) \( \frac{1}{2}(1-x)^{-1/2} \)
(b) \( -\frac{1}{2}(1-x)^{-1/2} \)
(c) \( -(1-x)^{-1/2} \)
(d) \( -\frac{1}{2}(1-x)^{1/2} \)
2.5.11 \( \frac{d}{dx} e^{2x} = \)

(a) \( x^2 e^{2x} \)
(b) \( x^2 e^{x^2} \)
(c) \( 2xe^{x^2} \)
(d) \( xe^{x^2} \)

2.5.12 \( \frac{d}{dx} 3^{4x+1} = \)

(a) \( 4 \cdot 3^{4x+1} \ln 4 \)
(b) \( 4 \cdot 3^{4x+1} \ln 3 \)
(c) \( (4x + 1) \cdot 3^{4x+1} \ln 3 \)
(d) \( (4x + 1) \cdot 3^{4x+1} \ln 4 \)

2.5.13 \( \frac{d}{dx} (e^x + x^2)^2 = \)

(a) \( 2 (e^x + x^2) \)
(b) \( 2 (e^x + 2x) (e^x + x^2)^2 \)
(c) \( 2 (e^x + x^2)^2 \)
(d) \( 2 (e^x + 2x) (e^x + x^2) \)

2.5.14 \( \frac{d}{dx} x^2 e^{-2x} = \)

(a) \( x^2 e^{-2x} - 2x^2 e^{-2x} \)
(b) \( 2xe^{-2x} - x^2 e^{-2x} \)
(c) \( 2xe^{-2x} - 2x^2 e^{-2x} \)
(d) \( -2x^2 e^{-2x} \)

2.5.15 \( \frac{d}{dx} 3^{e^{2x}} = \)

(a) \( 2e^{2x} 3^{e^{2x}} \ln 3 \)
(b) \( 2e^{2x} 3^{e^{2x}} \)
(c) \( 2 \cdot 3^{e^{2x}} \ln 3 \)
(d) \( e^{2x} 3^{e^{2x}} \ln 3 \)
(e) \( 2 \cdot 3^{e^{2x}} \)
2.5.16 If $\frac{dy}{dx} = 5$ and $\frac{dx}{dt} = -2$ then $\frac{dy}{dt} =$
(a) 5
(b) -2
(c) -10
(d) cannot be determined from the information given

2.5.17 If $\frac{dz}{dx} = 12$ and $\frac{dy}{dx} = 2$ then $\frac{dz}{dy} =$
(a) 24
(b) 6
(c) 1/6
(d) cannot be determined from the information given

2.5.18 If $y = 5x^2$ and $\frac{dx}{dt} = 3$, then when $x = 4, \frac{dy}{dx} =$
(a) 12
(b) 80
(c) 120
(d) $15x^2$
(e) cannot be determined from the information given

2.5.19 The functions $f(x)$ and $h(x)$ are plotted below. The function $g(x) = f(h(x))$. What is $g'(2)$?

(a) $g'(2) = -\frac{1}{2}$
(b) $g'(2) = 1$
(c) $g'(2) = 3$
(d) $g'(2) = 4$
(e) $g'(2)$ is undefined
2.6 Derivatives of Inverse Functions

Preview Activity 2.6. The equation \( y = \frac{5}{9}(x - 32) \) relates a temperature given in \( x \) degrees Fahrenheit to the corresponding temperature \( y \) measured in degrees Celsius.

(a) Solve the equation \( y = \frac{5}{9}(x - 32) \) for \( x \) to write \( x \) (Fahrenheit temperature) in terms of \( y \) (Celsius temperature).

(b) Let \( C(x) = \frac{5}{9}(x - 32) \) be the function that takes a Fahrenheit temperature as input and produces the Celsius temperature as output. In addition, let \( F(y) \) be the function that converts a temperature given in \( y \) degrees Celsius to the temperature \( F(y) \) measured in degrees Fahrenheit. Use your work in (a) to write a formula for \( F(y) \).

(c) Next consider the new function defined by \( p(x) = F(C(x)) \). Use the formulas for \( F \) and \( C \) to determine an expression for \( p(x) \) and simplify this expression as much as possible. What do you observe?

(d) Now, let \( r(y) = C(F(y)) \). Use the formulas for \( F \) and \( C \) to determine an expression for \( r(y) \) and simplify this expression as much as possible. What do you observe?

(e) What is the value of \( C'(x) \) of \( F'(y) \)? How do these values appear to be related?
Activity 2.16.

For each function given below, find its derivative.

(a) \( h(x) = x^2 \ln(x) \)

(b) \( p(t) = \frac{\ln(t)}{e^t + 1} \)

(c) \( s(y) = \ln(\cos(y) + 2) \)

(d) \( z(x) = \tan(\ln(x)) \)

(e) \( m(z) = \ln(\ln(z)) \)
Activity 2.17.

The following prompts in this activity will lead you to develop the derivative of the inverse tangent function.

(a) Let \( r(x) = \arctan(x) \). Use the relationship between the arctangent and tangent functions to rewrite this equation using only the tangent function.

(b) Differentiate both sides of the equation you found in (a). Solve the resulting equation for \( r'(x) \), writing \( r'(x) \) as simply as possible in terms of a trigonometric function evaluated at \( r(x) \).

(c) Recall that \( r(x) = \arctan(x) \). Update your expression for \( r'(x) \) so that it only involves trigonometric functions and the independent variable \( x \).

(d) Introduce a right triangle with angle \( \theta \) so that \( \theta = \arctan(x) \). What are the three sides of the triangle?

(e) In terms of only \( x \) and \( 1 \), what is the value of \( \cos(\arctan(x)) \)?

(f) Use the results of your work above to find an expression involving only \( 1 \) and \( x \) for \( r'(x) \).
Activity 2.18.

Determine the derivative of each of the following functions.

(a) \( f(x) = x^3 \arctan(x) + e^x \ln(x) \)
(b) \( p(t) = 2^t \arcsin(t) \)
(c) \( h(z) = (\arcsin(5z) + \arctan(4 - z))^27 \)
(d) \( s(y) = \cot(\arctan(y)) \)
(e) \( m(v) = \ln(\sin^2(v) + 1) \)
(f) \( g(w) = \arctan \left( \frac{\ln(w)}{1 + w^2} \right) \)
Voting Questions

2.6.1 \( \frac{d}{dt} \ln(t^2 + 1) \) is

(a) \( 2t \ln(t^2 + 1) \)
(b) \( \frac{2t}{t^2+1} \)
(c) \( \frac{dt}{\ln(t^2+1)} \)
(d) \( \frac{1}{t^2+1} \)

2.6.2 \( \frac{d}{dx} \ln(1 - x) \) is

(a) \( -\ln(1 - x) \)
(b) \( -2x(1 - x^2)^{-1} \)
(c) \( -(1 - x) \)
(d) \( -(1 - x)^{-1} \)

2.6.3 \( \frac{d}{dx} \ln(\pi) \) is

(a) \( \frac{1}{\pi} \)
(b) \( \frac{\ln(\pi)}{\pi} \)
(c) \( e^\pi \)
(d) \( 0 \)

2.6.4 \( \frac{d}{d\theta} \ln(\cos \theta) \) is

(a) \( \frac{\sin \theta}{\cos \theta} \)
(b) \( -\sin \theta \ln(\cos \theta) \)
(c) \( \frac{-\sin \theta}{\cos \theta} \)
(d) \( \frac{\sin \theta}{\ln(\cos \theta)} \)

2.6.5 Find \( f'(x) \) if \( f(x) = \log_5(2x + 1) \).

(a) \( f'(x) = \frac{2}{\ln 5} \cdot \frac{1}{2x + 1} \)
2.6. DERIVATIVES OF INVERSE FUNCTIONS

(b) \( f'(x) = \frac{2 \ln 5}{2x + 1} \)
(c) \( f'(x) = \frac{2}{\log_5(2x + 1)} \)
(d) \( f'(x) = \frac{2}{2x + 1} \)

2.6.6 If \( g(x) = \sin^{-1} x \), then \( g'(x) \) is

(a) \( \frac{1}{\sqrt{1-x^2}} \)
(b) \( \frac{1}{\cos x} \)
(c) \( -\frac{\cos x}{\sin^2 x} \)
(d) \( \csc x \cot x \)

2.6.7 If \( g(x) = (\sin x)^{-1} \), then \( g'(x) \) is

(a) \( \frac{1}{\sqrt{1-x^2}} \)
(b) \( \frac{1}{\cos x} \)
(c) \( -\frac{\cos x}{\sin^2 x} \)
(d) \( \csc x \cot x \)

2.6.8 If \( p(x) = 3 \ln(2x + 7) \), then \( p'(2) \) is

(a) \( \frac{6}{11} \)
(b) \( \frac{6}{2x+7} \)
(c) \( \frac{3}{2} \)
(d) \( \frac{3}{x} \)
(e) \( \frac{3}{11} \)

2.6.9 If \( q = a^2 \ln(a^3 \sin b + b^2 c) \), then \( \frac{dq}{db} \) is

(a) \( \frac{a^2}{a^3 \cos b + b^2 c} \)
(b) \( \frac{a^3 \cos b + 2a^2 bc}{a^3 \sin b + b^2 c} \)
(c) \( \frac{a^3 \cos b + 2bc}{a^3 \sin b + b^2 c} \)
(d) \( \frac{6a^3 \cos b + 4ab}{a^3 \cos b + b^2 c} \)
2.7 Derivatives of Functions Given Implicitly

Preview Activity 2.7. Let $f$ be a differentiable function of $x$ (whose formula is not known) and recall that $\frac{d}{dx}[f(x)]$ and $f'(x)$ are interchangeable notations. Determine each of the following derivatives of combinations of explicit functions of $x$, the unknown function $f$, and an arbitrary constant $c$.

(a) $\frac{d}{dx}[x^2 + f(x)]$

(b) $\frac{d}{dx}[x^2 f(x)]$

(c) $\frac{d}{dx}[c + x + f(x)^2]$

(d) $\frac{d}{dx}[f(x^2)]$

(e) $\frac{d}{dx}[xf(x) + f(cx) + cf(x)]$
Activity 2.19.

Consider the curve defined by the equation \( x = y^5 - 5y^3 + 4y \), whose graph is pictured in Figure 2.4.

(a) Explain why it is not possible to express \( y \) as an explicit function of \( x \).

(b) Use implicit differentiation to find a formula for \( \frac{dy}{dx} \).

(c) Use your result from part (b) to find an equation of the line tangent to the graph of \( x = y^5 - 5y^3 + 4y \) at the point (0, 1).

(d) Use your result from part (b) to determine all of the points at which the graph of \( x = y^5 - 5y^3 + 4y \) has a vertical tangent line.

\[\text{Figure 2.4: The curve } x = y^5 - 5y^3 + 4y.\]
Activity 2.20.

Consider the curve defined by the equation \( y(y^2 - 1)(y - 2) = x(x - 1)(x - 2) \), whose graph is pictured in Figure 2.5. Through implicit differentiation, it can be shown that

\[
\frac{dy}{dx} = \frac{(x-1)(x-2) + x(x-2) + x(x-1)}{(y^2 - 1)(y-2) + 2y^2(y-2) + y(y^2 - 1)}.
\]

Use this fact to answer each of the following questions.

(a) Determine all points \((x, y)\) at which the tangent line to the curve is horizontal.

(b) Determine all points \((x, y)\) at which the tangent line is vertical.

(c) Find the equation of the tangent line to the curve at one of the points where \(x = 1\).
2.7. DERIVATIVES OF FUNCTIONS GIVEN IMPLICITLY

Activity 2.21.

For each of the following curves, use implicit differentiation to find \( \frac{dy}{dx} \) and determine the equation of the tangent line at the given point.

(a) \( x^3 - y^3 = 6xy, \quad (-3, 3) \)
(b) \( \sin(y) + y = x^3 + x, \quad (0, 0) \)
(c) \( xe^{-xy} = y^2, \quad (0.571433, 1) \)
Voting Questions

2.7.1 Find $\frac{dy}{dx}$ implicitly if $y^3 = x^2 + 1$.

(a) $\frac{dy}{dx} = \frac{2}{3}x$
(b) $\frac{dy}{dx} = 0$
(c) $\frac{dy}{dx} = \frac{x^2 + 1}{3y^2}$
(d) $\frac{dy}{dx} = \frac{2x}{3y^2}$
Preview Activity 2.8. Let \( h \) be the function given by \( h(x) = \frac{x^5 + x - 2}{x^2 - 1} \).

(a) What is the domain of \( h \)?

(b) Explain why \( \lim_{x \to 1} \frac{x^5 + x - 2}{x^2 - 1} \) results in an indeterminate form.

(c) Next we will investigate the behavior of both the numerator and denominator of \( h \) near the point where \( x = 1 \). Let \( f(x) = x^5 + x - 2 \) and \( g(x) = x^2 - 1 \). Find the local linearizations of \( f \) and \( g \) at \( a = 1 \), and call these functions \( L_f(x) \) and \( L_g(x) \), respectively.

(d) Explain why \( h(x) \approx \frac{L_f(x)}{L_g(x)} \) for \( x \) near \( a = 1 \).

(e) Using your work from (c), evaluate 
\[
\lim_{x \to 1} \frac{L_f(x)}{L_g(x)}
\]

What do you think your result tells us about \( \lim_{x \to 1} h(x) \)?

(f) Investigate the function \( h(x) \) graphically and numerically near \( x = 1 \). What do you think is the value of \( \lim_{x \to 1} h(x) \)?
Activity 2.22.
Evaluate each of the following limits. If you use L’Hopital’s Rule, indicate where it was used, and be certain its hypotheses are met before you apply it.

(a) \( \lim_{x \to 0} \frac{\ln(1 + x)}{x} \)

(b) \( \lim_{x \to \pi} \frac{\cos(x)}{x} \)

(c) \( \lim_{x \to 1} \frac{2 \ln(x)}{1 - e^{x-1}} \)

(d) \( \lim_{x \to 0} \frac{\sin(x) - x}{\cos(2x) - 1} \)
Activity 2.23.

In this activity, we reason graphically to evaluate limits of ratios of functions about which some information is known.

![Graphs](image)

Figure 2.6: Three graphs referenced in the questions of Activity 2.23.

(a) Use the left-hand graph to determine the values of \( f(2) \), \( f'(2) \), \( g(2) \), and \( g'(2) \). Then, evaluate

\[
\lim_{x \to 2} \frac{f(x)}{g(x)}.
\]

(b) Use the middle graph to find \( p(2) \), \( p'(2) \), \( q(2) \), and \( q'(2) \). Then, determine the value of

\[
\lim_{x \to 2} \frac{p(x)}{q(x)}.
\]

(c) Use the right-hand graph to compute \( r(2) \), \( r'(2) \), \( s(2) \), \( s'(2) \). Explain why you cannot determine the exact value of

\[
\lim_{x \to 2} \frac{r(x)}{s(x)}
\]

without further information being provided, but that you can determine the sign of \( \lim_{x \to 2} \frac{r(x)}{s(x)} \). In addition, state what the sign of the limit will be, with justification.
Activity 2.24.

Evaluate each of the following limits. If you use L’Hopital’s Rule, indicate where it was used, and be certain its hypotheses are met before you apply it.

(a) \( \lim_{x \to \infty} \frac{x}{\ln(x)} \)

(b) \( \lim_{x \to \infty} \frac{e^x + x}{2e^x + x^2} \)

(c) \( \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} \)

(d) \( \lim_{x \to \frac{\pi}{2}} \frac{\tan(x)}{x - \frac{\pi}{2}} \)

(e) \( \lim_{x \to \infty} xe^{-x} \)
2.8. USING DERIVATIVES TO EVALUATE LIMITS

Voting Questions

2.8.1 Evaluate \( \lim_{x \to \pi/2^-} \frac{1 + \tan x}{\sec x} \).

(a) 0
(b) 1
(c) \( \infty \)
(d) \(-\infty \)

Other Voting Questions for Derivatives

2.8.2 A function intersects the \( x \)-axis at points \( a \) and \( b \), where \( a < b \). The slope of the function at \( a \) is positive and the slope at \( b \) is negative. Which of the following is true for any such function? There exists some point on the interval \((a, b)\) where

(a) the slope is zero and the function has a local maximum.
(b) the slope is zero but there is not a local maximum.
(c) there is a local maximum, but there doesn’t have to be a point at which the slope is zero.
(d) none of the above have to be true.

2.8.3 A function intersects the \( x \)-axis at points \( a \) and \( b \), where \( a < b \). The slope of the function at \( a \) is positive and the slope at \( b \) is negative. Which of the following is true for any such function for which the limit of the function exists and is finite at every point? There exists some point on the interval \((a, b)\) where

(a) the slope is zero and the function has a local maximum.
(b) the slope is zero but there is not a local maximum.
(c) there is a local maximum, but there doesn’t have to be a point at which the slope is zero.
(d) none of the above have to be true.

2.8.4 A continuous function intersects the \( x \)-axis at points \( a \) and \( b \), where \( a < b \). The slope of the function at \( a \) is positive and the slope at \( b \) is negative. Which of the following is true for any such function? There exists some point on the interval \((a, b)\) where

(a) the slope is zero and the function has a local maximum.
(b) the slope is zero but there is not a local maximum.
(c) there is a local maximum, but there doesn't have to be a point at which the slope is zero.
(d) none of the above have to be true.

2.8.5 A continuous and differentiable function intersects the $x$-axis at points $a$ and $b$, where $a < b$. The slope of the function at $a$ is positive and the slope at $b$ is negative. Which of the following is true for any such function? There exists some point on the interval $(a, b)$ where

(a) the slope is zero and the function has a local maximum.
(b) the slope is zero but there is not a local maximum.
(c) there is a local maximum, but there doesn't have to be a point at which the slope is zero.
(d) none of the above have to be true.

2.8.6 On a toll road a driver takes a time stamped toll-card from the starting booth and drives directly to the end of the toll section. After paying the required toll, the driver is surprised to receive a speeding ticket along with the toll receipt. Which of the following best describes the situation?

(a) The booth attendant does not have enough information to prove that the driver was speeding.
(b) The booth attendant can prove that the driver was speeding during his trip.
(c) The driver will get a ticket for a lower speed than his actual maximum speed.
(d) Both (b) and (c).

2.8.7 True or False: For $f(x) = |x|$ on the interval $[-\frac{1}{2}, 2]$, you can find a point $c$ in $(-\frac{1}{2}, 2)$, such that $f'(c) = \frac{f(2) - f(-\frac{1}{2})}{2 - (-\frac{1}{2})}$.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.8.8 A racer is running back and forth along a straight path. He finishes the race at the place where he began.

True or False: There had to be at least one moment, other than the beginning and the end of the race, when he “stopped” (i.e., his speed was 0).
2.8. USING DERIVATIVES TO EVALUATE LIMITS

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.8.9 Two racers start a race at the same moment and finish in a tie. Which of the following must be true?

(a) At some point during the race the two racers were not tied.
(b) The racers’ speeds at the end of the race must have been exactly the same.
(c) The racers must have had the same speed at exactly the same time at some point in the race.
(d) The racers had to have the same speed at some moment, but not necessarily at exactly the same time.

2.8.10 Two horses start a race at the same time and one runs slower than the other throughout the race. **True or False:** The Racetrack Principle can be used to justify the fact that the slower horse loses the race.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.8.11 **True or False:** The Racetrack Principle can be used to justify the statement that if two horses start a race at the same time, the horse that wins must have been moving faster than the other throughout the race.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

2.8.12 Which of the following statements illustrates a correct use of the Racetrack Principle?

(a) Since \( \sin 0 = 0 \) and \( \cos x \leq 1 \) for all \( x \), the Racetrack Principle tells us that \( \sin x \leq x \) for all \( x \geq 0 \).
(b) For $a < b$, if $f'(x)$ is positive on $[a, b]$ then the Racetrack Principle tells us that $f(a) < f(b)$.

(c) Let $f(x) = x$ and $g(x) = x^2 - 2$. Since $f(-1) = g(-1) = -1$ and $f(1) > g(1)$, the Racetrack Principle tells us that $f'(x) > g'(x)$ for $-1 < x < 1$.

(d) All are correct uses of the Racetrack Principle.

(e) Exactly 2 of a, b, and c are correct uses of the Racetrack Principle.
2.9 More on Limits

Preview Activity 2.9. Evaluate the following limits.

(a) \( \lim_{x \to -1} \frac{x^2 - x - 2}{x^3 - x} \)
(b) \( \lim_{x \to 1} \frac{x^2 - x - 2}{x^3 - x} \)
(c) \( \lim_{x \to 2} \arcsin(x^2 - 5) \)
(d) \( \lim_{x \to 0} \frac{\sin 3x}{4x} \)
(e) \( \lim_{x \to 2} \frac{|x^2 + x - 6|}{x - 2} \)
(f) \( \lim_{x \to 0^+} x \ln x^4 \)
Preview Activity 2.10. Write an equivalent expression for each expression below by multiplying and dividing by the conjugate. Simplify your answer as much as possible.

(a) \( \frac{\sqrt{x} - 1}{x - 1} \)

(b) \( \frac{x - 16}{\sqrt{x} - 4} \)

(c) \( \frac{x}{\sqrt{7 + x} - \sqrt{7}} \)

(d) \( \sqrt{9x^2 + x - 3x} \)

Use your work to evaluate the following limits.

(a) \( \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \)

(b) \( \lim_{x \to 16} \frac{x - 16}{\sqrt{x} - 4} \)

(c) \( \lim_{x \to 0} \frac{x}{\sqrt{7 + x} - \sqrt{7}} \)

(d) \( \lim_{x \to \infty} \sqrt{9x^2 + x - 3x} \)
Activity 2.25.

In this exercise we wish to graph the curve $y = \sqrt{4x^2 - 1}$.

(a) Find the domain of the function $f(x) = \sqrt{4x^2 - 1}$. Notice that some negative values of $x$ are allowed in the domain.

(b) Evaluate the derivative of $f(x)$ and find the intervals where the function is increasing and decreasing.

(c) Evaluate the limit $\lim_{x \to \infty} \sqrt{4x^2 - 1 - 2x}$.

(d) If $y = ax + b$ is a slant asymptote of $y = f(x)$, then the values of $ax + b$ and $f(x)$ are getting closer for increasingly large values of $x$ (we only discuss the case were $x \to \infty$, but the case where $x \to -\infty$ is similar). In other words, $y = ax + b$ is a slant asymptote of $y = f(x)$ if

$$\lim_{x \to \infty} f(x) - (ax + b) = 0.$$ 

Use the limit in (c) to find a slant asymptote for the graph of $y = f(x)$.

![Figure 2.7](image-url)

Figure 2.7: This figure illustrates a function $f$ which has a slant asymptote $y = ax + b$.

(e) When $x$ is large and positive, $4x^2$ is much larger than 1, and we may write $\sqrt{4x^2 - 1} \approx \sqrt{4x^2}$. Simplify $\sqrt{4x^2}$ to obtain another argument showing that $y = 2x$ is a slant asymptote of the graph of $y = f(x)$. How would this argument work out for large but negative values of $x$? Hint: $\sqrt{4x^2}$ is not equal to $2x$ when $x$ is negative. What is it equal to?

(f) Graph $y = f(x)$ for nonnegative values of $x$ using information on the domain, the derivative and the asymptote.

(g) Show $f(-1) = f(1)$, $f(-2.4) = f(2.4)$, and in general that $f(-x) = f(x)$ to argue that $f$ is symmetrical with respect to the $y$-axis. Use your previous graph to obtain the graph of $f$ over its domain.

(h) Consider the curve $y = \sqrt{9x^2 + x - 1} + 1$ and find its slant asymptotes.
Activity 2.26.

Evaluate the following limits.

(a) \( \lim_{x \to 2} \cos \left( \frac{\pi (x^2 - 4)}{x - 2} \right) \)

(b) \( \lim_{x \to \infty} \sqrt{\frac{2x^2 + x - 1}{3x^2 - 4}} \)

(c) \( \lim_{x \to 9} e^{\frac{\sqrt{x} - 3}{x - 9}} \)

(d) \( \lim_{x \to \infty} \arcsin \left( \frac{\sqrt{3} + x}{2x} \right) \)
Activity 2.27.

Find all values where the following functions are continuous.

(a) \( \sqrt{1 - x^2} \)

(b) \( x^2 + 4 + \frac{1}{\sqrt{1 - x^2}} \)

(c) \( \cos \left( e^{1/x} \right) \)

(d) \( \arctan \left( \frac{x^2 - 4}{3x^2 - 6x} \right) \)
Activity 2.28.
Evaluate the following limits

(a) \[ \lim_{x \to 0^+} \sqrt{x} \sin \left( \frac{1}{x^2} \right) \]

(b) \[ \lim_{x \to \infty} \sqrt{x} \arctan \left( \frac{1}{x} \right) \] (hint: \(0 \leq \arctan z \leq z\) for \(z \geq 0\) and notice that when \(x\) is positive, \(1/x\) is also positive.)

(c) Define for an integer \(n\) the factorial of \(n\) written as \(n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n\). For instance, \(4! = 1 \times 2 \times 3 \times 4 = 24\). Use the Squeeze Theorem to evaluate the following limit

\[ \lim_{n \to \infty} \frac{2^n}{n!}. \]

Here are a few steps you may want to follow:

1. Consider \(n\) as a large positive integer and argue that

\[ \frac{2^n}{n!} = \left( \frac{2}{1} \right) \cdot \left( \frac{2}{2} \right) \cdot \left( \frac{2}{3} \right) \cdot \left( \frac{2}{4} \right) \ldots \left( \frac{2}{n} \right). \]

2. Use the previous equality and the fact that \(2/i \leq 2/3\) for \(i \geq 3\) to show that

\[ 0 \leq \frac{2^n}{n!} \leq 2 \cdot \left( \frac{2}{3} \right)^{n-2}. \]

3. Use the Squeeze Theorem to find

\[ \lim_{n \to \infty} \frac{2^n}{n!}. \]

4. Which function dominates as \(n \to \infty\): the exponential function \(2^n\) or the factorial function \(n!\)?

(d) In Appendix B, read the proof for the differentiation formula

\[ \frac{d}{dx} [\sin(x)] = \cos(x) \]

and identify where the Squeeze Theorem is used.
Chapter 3

Using Derivatives

3.1 Using derivatives to identify extreme values of a function

Preview Activity 3.1. Consider the function \( h \) given by the graph in Figure 3.1. Use the graph to answer each of the following questions.

(a) Identify all of the values of \( c \) for which \( h(c) \) is a local maximum of \( h \).
(b) Identify all of the values of \( c \) for which \( h(c) \) is a local minimum of \( h \).
(c) Does \( h \) have a global maximum? If so, what is the value of this global maximum?
(d) Does \( h \) have a global minimum? If so, what is its value?
(e) Identify all values of \( c \) for which \( h'(c) = 0 \).

Figure 3.1: The graph of a function \( h \) on the interval \([-3, 3]\).
(f) Identify all values of $c$ for which $h'(c)$ does not exist.

(g) True or false: every relative maximum and minimum of $h$ occurs at a point where $h'(c)$ is either zero or does not exist.

(h) True or false: at every point where $h'(c)$ is zero or does not exist, $h$ has a relative maximum or minimum.

\[ \boxed{} \]
Activity 3.1.
Suppose that \( g(x) \) is a function continuous for every value of \( x \neq 2 \) whose first derivative is \( g'(x) = \frac{(x + 4)(x - 1)^2}{x - 2} \). Further, assume that it is known that \( g \) has a vertical asymptote at \( x = 2 \).

(a) Determine all critical values of \( g \).

(b) By developing a carefully labeled first derivative sign chart, decide whether \( g \) has as a local maximum, local minimum, or neither at each critical value.

(c) Does \( g \) have a global maximum? global minimum? Justify your claims.

(d) What is the value of \( \lim_{x \to \infty} g'(x) \)? What does the value of this limit tell you about the long-term behavior of \( g \)?

(e) Sketch a possible graph of \( y = g(x) \).
Activity 3.2.

Suppose that \( g \) is a function whose second derivative, \( g'' \), is given by the following graph.

![Graph of \( g''(x) \)](image)

Figure 3.2: The graph of \( y = g''(x) \).

(a) Find all points of inflection of \( g \).

(b) Fully describe the concavity of \( g \) by making an appropriate sign chart.

(c) Suppose you are given that \( g'(-1.67857351) = 0 \). Is there a local maximum, local minimum, or neither (for the function \( g \)) at this critical value of \( g \), or is it impossible to say? Why?

(d) Assuming that \( g''(x) \) is a polynomial (and that all important behavior of \( g'' \) is seen in the graph above, what degree polynomial do you think \( g(x) \) is? Why?
Activity 3.3.

Consider the family of functions given by $h(x) = x^2 + \cos(kx)$, where $k$ is an arbitrary positive real number.

(a) Use a graphing utility to sketch the graph of $h$ for several different $k$-values, including $k = 1, 3, 5, 10$. Plot $h(x) = x^2 + \cos(3x)$ on the axes provided below. What is the smallest value of $k$ at which you think you can see (just by looking at the graph) at least one inflection point on the graph of $h$?

(b) Explain why the graph of $h$ has no inflection points if $k \leq \sqrt{2}$, but infinitely many inflection points if $k > \sqrt{2}$.

(c) Explain why, no matter the value of $k$, $h$ can only have a finite number of critical values.

Figure 3.3: Axes for plotting $y = h(x)$. 
Voting Questions

3.1.1 If \( f''(a) = 0 \), then \( f \) has an inflection point at \( a \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.1.2 A local maximum of \( f \) only occurs at a point where \( f'(x) = 0 \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.1.3 If \( x = p \) is not a local minimum or maximum of \( f \), then \( x = p \) is not a critical point of \( f \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.1.4 If \( f'(x) \) is continuous and \( f(x) \) has no critical points, then \( f \) is everywhere increasing or everywhere decreasing.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.1.5 If \( f'(x) \geq 0 \) for all \( x \), then \( f(a) \leq f(b) \) whenever \( a \leq b \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
3.1. USING DERIVATIVES TO IDENTIFY EXTREME VALUES OF A FUNCTION

(d) False, and I am very confident

3.1.6 Imagine that you are skydiving. The graph of your speed as a function of time from the time you jumped out of the plane to the time you achieved terminal velocity is

(a) increasing and concave up
(b) decreasing and concave up
(c) increasing and concave down
(d) decreasing and concave down

3.1.7 Water is being poured into a “Dixie cup” (a standard cup that is smaller at the bottom than at the top). The height of the water in the cup is a function of the volume of water in the cup. The graph of this function is

(a) increasing and concave up
(b) increasing and concave down
(c) a straight line with positive slope.
3.2 Using derivatives to describe families of functions

Preview Activity 3.2. Let $a$, $h$, and $k$ be arbitrary real numbers with $a \neq 0$, and let $f$ be the function given by the rule $f(x) = a(x - h)^2 + k$. See http://www.geogebratube.org/student/m68215 to explore the effect that $a$, $h$, and $k$ have on this function.

(a) What familiar type of function is $f$? What information do you know about $f$ just by looking at its form? (Think about the roles of $a$, $h$, and $k$.)

(b) Next we use some calculus to develop familiar ideas from a different perspective. To start, treat $a$, $h$, and $k$ as constants and compute $f'(x)$.

(c) Find all critical values of $f$. (These will depend on at least one of $a$, $h$, and $k$.)

(d) Assume that $a < 0$. Construct a first derivative sign chart for $f$.

(e) Based on the information you’ve found above, classify the critical values of $f$ as maxima or minima.

\[\square\]
Activity 3.4.

Consider the family of functions defined by \( p(x) = x^3 - ax \), where \( a \neq 0 \) is an arbitrary constant. See the geogebra applet http://www.geogebratube.org/student/m110845.

(a) Find \( p'(x) \) and determine the critical values of \( p \). How many critical values does \( p \) have?

(b) Construct a first derivative sign chart for \( p \). What can you say about the overall behavior of \( p \) if the constant \( a \) is positive? Why? What if the constant \( a \) is negative? In each case, describe the relative extremes of \( p \).

(c) Find \( p''(x) \) and construct a second derivative sign chart for \( p \). What does this tell you about the concavity of \( p \)? What role does \( a \) play in determining the concavity of \( p \)?

(d) Without using a graphing utility, sketch and label typical graphs of \( p(x) \) for the cases where \( a > 0 \) and \( a < 0 \). Label all inflection points and local extrema.

(e) Finally, use a graphing utility to test your observations above by entering and plotting the function \( p(x) = x^3 - ax \) for at least four different values of \( a \). Write several sentences to describe your overall conclusions about how the behavior of \( p \) depends on \( a \).
Activity 3.5.

Consider the two-parameter family of functions of the form \( h(x) = a(1 - e^{-bx}) \), where \( a \) and \( b \) are positive real numbers.

(a) Find the first derivative and the critical values of \( h \). Use these to construct a first derivative sign chart and determine for which values of \( x \) the function \( h \) is increasing and decreasing.

(b) Find the second derivative and build a second derivative sign chart. For which values of \( x \) is a function in this family concave up? concave down?

(c) What is the value of \( \lim_{x \to \infty} a(1 - e^{-bx}) \)? \( \lim_{x \to -\infty} a(1 - e^{-bx}) \)?

(d) How does changing the value of \( b \) affect the shape of the curve?

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function \( h \) and how this behavior depends on \( a \) and \( b \).
Activity 3.6.

Let \( L(t) = \frac{A}{1 + ce^{-kt}} \), where \( A, c, \) and \( k \) are all positive real numbers.

(a) Observe that we can equivalently write \( L(t) = A(1 + ce^{-kt})^{-1} \). Find \( L'(t) \) and explain why \( L \) has no critical values. Is \( L \) always increasing or always decreasing? Why?

(b) Given the fact that

\[
L''(t) = Ak^2e^{-kt} \frac{ce^{-kt} - 1}{(1 + ce^{-kt})^3},
\]

find all values of \( t \) such that \( L''(t) = 0 \) and hence construct a second derivative sign chart. For which values of \( t \) is a function in this family concave up? concave down?

(c) What is the value of \( \lim_{t \to \infty} \frac{A}{1 + ce^{-kt}}? \lim_{t \to -\infty} \frac{A}{1 + ce^{-kt}}? \)

(d) Find the value of \( L(x) \) at the inflection point found in (b).

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function \( h \) and how this behavior depends on \( a \) and \( b \).

(f) Explain why it is reasonable to think that the function \( L(t) \) models the growth of a population over time in a setting where the largest possible population the surrounding environment can support is \( A \).
3.2.1 The functions in the figure have the form \( y = A \sin x \). Which of the functions has the largest \( A \)? Assume the scale on the vertical axes is the same for each graph.

3.2.2 The functions in the figure have the form \( y = \sin(Bx) \). Which of the functions has the largest \( B \)? Assume the scale on the horizontal axes is the same for each graph.

3.2.3 Let \( f(x) = ax + \frac{b}{x} \). What are the critical points of \( f(x) \)?
   (a) \(-\frac{b}{a}\)  (b) 0  (c) \(\pm \sqrt{b/a}\)  (d) \(\pm \sqrt{-b/a}\)  (e) No critical points

3.2.4 Let \( f(x) = ax + \frac{b}{x} \). Suppose \( a \) and \( b \) are positive. What happens to \( f(x) \) as \( b \) increases?
   (a) The critical points move further apart.
   (b) The critical points move closer together.

3.2.5 Let \( f(x) = ax + \frac{b}{x} \). Suppose \( a \) and \( b \) are positive. What happens to \( f(x) \) as \( b \) increases?
   (a) The critical values move further apart.
   (b) The critical values move closer together.
3.2.6 Let \( f(x) = ax + b/x \). Suppose \( a \) and \( b \) are positive. What happens to \( f(x) \) as \( a \) increases?

(a) The critical points move further apart.
(b) The critical points move closer together.

3.2.7 Let \( f(x) = ax + b/x \). Suppose \( a \) and \( b \) are positive. What happens to \( f(x) \) as \( a \) increases?

(a) The critical values move further apart.
(b) The critical values move closer together.

3.2.8 Find a formula for a parabola with its vertex at (3,2) and with a second derivative of -4.

(a) \( y = -4x^2 + 48x - 106 \).
(b) \( y = -4x^2 + 24x - 34 \).
(c) \( y = -2x^2 + 12x - 16 \).
(d) \( y = -2x^2 + 4x + 8 \).
3.3 Global Optimization

Preview Activity 3.3. Let \( f(x) = 2 + \frac{3}{1 + (x + 1)^2} \).

(a) Determine all of the critical values of \( f \).

(b) Construct a first derivative sign chart for \( f \) and thus determine all intervals on which \( f \) is increasing or decreasing.

(c) Does \( f \) have a global maximum? If so, why, and what is its value and where is the maximum attained? If not, explain why.

(d) Determine \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

(e) Explain why \( f(x) > 2 \) for every value of \( x \).

(f) Does \( f \) have a global minimum? If so, why, and what is its value and where is the minimum attained? If not, explain why.
Activity 3.7.

Let \( g(x) = \frac{1}{3}x^3 - 2x + 2 \).

(a) Find all critical values of \( g \) that lie in the interval \(-2 \leq x \leq 3\).

(b) Use a graphing utility to construct the graph of \( g \) on the interval \(-2 \leq x \leq 3\).

(c) From the graph, determine the \( x \)-values at which the global minimum and global maximum of \( g \) occur on the interval \([-2, 3]\).

(d) How do your answers change if we instead consider the interval \(-2 \leq x \leq 2\)?

(e) What if we instead consider the interval \(-2 \leq x \leq 1\)?
Activity 3.8.

Find the exact absolute maximum and minimum of each function on the stated interval.

(a) \( h(x) = xe^{-x}, [0, 3] \)
(b) \( p(t) = \sin(t) + \cos(t), [-\frac{\pi}{2}, \frac{\pi}{2}] \)
(c) \( q(x) = \frac{x^2}{x-2}, [3, 7] \)
(d) \( f(x) = 4 - e^{-(x-2)^2}, (-\infty, \infty) \)
Activity 3.9.

A piece of cardboard that is 10 × 15 (each measured in inches) is being made into a box without a top. To do so, squares are cut from each corner of the box and the remaining sides are folded up. If the box needs to be at least 1 inch deep and no more than 3 inches deep, what is the maximum possible volume of the box? what is the minimum volume? Justify your answers using calculus.

(a) Draw a labeled diagram that shows the given information. What variable should we introduce to represent the choice we make in creating the box? Label the diagram appropriately with the variable, and write a sentence to state what the variable represents. (GeoGebra applet for the box folding problem)

(b) Determine a formula for the function V (that depends on the variable in (a)) that tells us the volume of the box.

(c) What is the domain of the function V? That is, what values of x make sense for input? Are there additional restrictions provided in the problem?

(d) Determine all critical values of the function V.

(e) Evaluate V at each of the endpoints of the domain and at any critical values that lie in the domain.

(f) What is the maximum possible volume of the box? the minimum?
Voting Questions

3.3.1 If \( f(x) \) is continuous on a closed interval, then it is enough to look at the points where \( f'(x) = 0 \) in order to find its global maxima and minima.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.3.2 A function defined on all points of a closed interval always has a global maximum and a global minimum.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.3.3 Let \( f \) be a continuous function on the closed interval \( 0 \leq x \leq 1 \). There exists a positive number \( A \) so that the graph of \( f \) can be drawn inside the rectangle \( 0 \leq x \leq 1, -A \leq y \leq A \).

The above statement is:

(a) Always true.
(b) Sometimes true.
(c) Not enough information.

3.3.4 Let \( f(x) = x^2 \). \( f \) has an upper bound on the interval \( (0, 2) \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.3.5 Let \( f(x) = x^2 \). \( f \) has a global maximum on the interval \( (0, 2) \).

(a) True, and I am very confident
3.3. GLOBAL OPTIMIZATION

(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.3.6 Let \( f(x) = x^2 \). \( f \) has a global minimum on the interval \((0, 2)\).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.3.7 Let \( f(x) = x^2 \). \( f \) has a global minimum on any interval \([a, b]\).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.3.8 Consider \( f(x) = -3x^2 + 12x + 7 \) on the interval \(-2 \leq x \leq 4\). Where does this function have its global maximum value?

(a) \( x = -2 \) (b) \( x = 0 \) (c) \( x = 2 \) (d) \( x = 4 \)

3.3.9 Consider \( f(x) = -3x^2 + 12x + 7 \) on the interval \(-2 \leq x \leq 4\). Where does this function have its global minimum value?

(a) \( x = -2 \) (b) \( x = 0 \) (c) \( x = 2 \) (d) \( x = 4 \)
3.4 Introduction to Sensitivity Analysis

Preview Activity 3.4. Consider the functions below. Each is defined with parameters that may not be known exactly. Find an expression for the critical point(s) for each function.

(a) \( f(x) = x^3 - ax \)
(b) \( f(N) = \frac{N}{1 + kN^2} \)
(c) \( f(z) = z e^{-z^2/\sigma} \)
(d) \( f(t) = a(1 - e^{-bt}) \)
Activity 3.10.
Consider the family of functions defined by

\[ f(x) = \frac{x}{1 + kx^2} \]

where \( k > 0 \) is an arbitrary constant. The first and second derivatives are

\[ f'(x) = \frac{1 - kx^2}{(1 + kx^2)^2} \quad \text{and} \quad f''(x) = \frac{2kx(kx^2 - 3)}{(1 + kx^2)^3} \]

(a) Determine the critical values of \( f \) in terms of \( k \). How many critical values does \( f \) have?

(b) Construct a first derivative sign chart for \( f \) and discuss the behavior of the critical points.

(c) From part (a) you should have found two critical values. Write these values as functions of \( k \) and find the derivatives with respect to \( k \).

(d) Based on your answers to part (c) describe the sensitivity of the critical values of \( f \) with respect to the parameter \( k \). Create plots for critical values vs. \( k \) and the derivatives of the critical values vs. \( k \).
3.4. INTRODUCTION TO SENSITIVITY ANALYSIS

Voting Questions

3.4.1 What are the critical points of the function \( f(x) = x^3 - ax \)?

(a) \( x = 0, \pm \sqrt{a} \)
(b) \( x = \pm \sqrt{a/3} \)
(c) \( x = \pm \sqrt{3a} \)
(d) \( x = \pm \sqrt{-a/3} \)
(e) \( x = \pm \sqrt{-3a} \)
(f) No critical points exist.

3.4.2 Consider the critical points of the function \( f(x) = x^3 - ax \). Are these critical points more sensitive to large values of \( a \) or to small values of \( a \)?

(a) Slight changes to large values of \( a \) affect the critical points more.
(b) Slight changes to small values of \( a \) affect the critical points more.
(c) Slight changes to \( a \) affect the critical points equally, whether \( a \) is large or small.

3.4.3 We have a system where the critical points are as follows: \( x = \pm 5A^2/B \). Will the critical points be more sensitive to changes in \( A \) when \( A \) is small, or when \( A \) is large?

(a) The critical points will be more sensitive to changes in \( A \) when \( A \) is small.
(b) The critical points will be more sensitive to changes in \( A \) when \( A \) is large.
(c) The critical points will be equally affected by changes in \( A \), no matter whether \( A \) is small or large.

3.4.4 What are the critical points of the function \( f(z) = ze^{-z^2/\sigma} \)?

(a) \( z = 0 \)
(b) \( z = \pm \sqrt{\frac{\sigma}{2}} \)
(c) \( z = \pm \sqrt{1/2} \)
(d) \( z = \pm \sqrt{\sigma} \)
(e) \( z = \frac{\sigma}{2} \)
(f) No critical points exist.
3.4.5 Consider the critical points of the function \( f(z) = z e^{-z^2/\sigma} \). Are these critical points more sensitive to large or small values of \( \sigma \)?

(a) The critical points are more sensitive to small values of \( \sigma \).
(b) The critical points are more sensitive to large values of \( \sigma \).
(c) The critical points are equally sensitive to large and small values of \( \sigma \).

3.4.6 What are the critical points of the function \( f(N) = \frac{N}{1+kN^2} \)?

(a) \( N = \sqrt{\frac{1}{k}} \)
(b) \( N = \sqrt{k} \)
(c) \( N = \pm \sqrt{\frac{1}{k}} \)
(d) \( N = \pm \sqrt{\frac{1}{2k}} \)
(e) No critical points exist.

3.4.7 Consider the critical points of the function \( f(N) = \frac{N}{1+kN^2} \). Will the critical points be more sensitive to changes in \( k \) when \( k \) is large or when \( k \) is small?

(a) Critical points will be more sensitive to changes in \( k \) when \( k \) is large.
(b) Critical points will be more sensitive to changes in \( k \) when \( k \) is small.
(c) Critical points will be equally affected by changes in \( k \) no matter whether \( k \) is small or large.

3.4.8 What are the critical points of the function \( f(t) = a(1 - e^{-bt}) \)?

(a) \( t = 0 \)
(b) \( t = \pm \sqrt{\frac{b}{a}} \)
(c) \( t = -\frac{1}{b} \)
(d) \( t = -\frac{1}{b} \ln \frac{1}{ab} \)
(e) No critical points exist.
3.5 Applied Optimization

Preview Activity 3.5. According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where by “girth” we mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?

Figure 3.4: A rectangular parcel with a square end.

(a) Let \( x \) represent the length of one side of the square end and \( y \) the length of the longer side. Label these quantities appropriately on the image shown in Figure 3.4.

(b) What is the quantity to be optimized in this problem? Find a formula for this quantity in terms of \( x \) and \( y \).

(c) The problem statement tells us that the parcel’s girth plus length may not exceed 108 inches. In order to maximize volume, we assume that we will actually need the girth plus length to equal 108 inches. What equation does this produce involving \( x \) and \( y \)?

(d) Solve the equation you found in (c) for one of \( x \) or \( y \) (whichever is easier).

(e) Now use your work in (b) and (d) to determine a formula for the volume of the parcel so that this formula is a function of a single variable.

(f) Over what domain should we consider this function? Note that both \( x \) and \( y \) must be positive; how does the constraint that girth plus length is 108 inches produce intervals of possible values for \( x \) and \( y \)?

(g) Find the absolute maximum of the volume of the parcel on the domain you established in (f) and hence also determine the dimensions of the box of greatest volume. Justify that you’ve found the maximum using calculus.
Activity 3.11.

A soup can in the shape of a right circular cylinder is to be made from two materials. The material for the side of the can costs $0.015 per square inch and the material for the lids costs $0.027 per square inch. Suppose that we desire to construct a can that has a volume of 16 cubic inches. What dimensions minimize the cost of the can?

(a) Draw a picture of the can and label its dimensions with appropriate variables.

(b) Use your variables to determine expressions for the volume, surface area, and cost of the can.

(c) Determine the total cost function as a function of a single variable. What is the domain on which you should consider this function?

(d) Find the absolute minimum cost and the dimensions that produce this value.

(e) How sensitive is your solution in part (d) to the cost of the lid? In other words, if the cost of the lid were to change by a small amount would that cause large or small changes in the minimum cost?
Activity 3.12.

A hiker starting at a point $P$ on a straight road walks east towards point $Q$, which is on the road and 3 kilometers from point $P$. Two kilometers due north of point $Q$ is a cabin. The hiker will walk down the road for a while, at a pace of 8 kilometers per hour. At some point $Z$ between $P$ and $Q$, the hiker leaves the road and makes a straight line towards the cabin through the woods, hiking at a pace of 3 kph, as pictured in Figure 3.5. In order to minimize the time to go from $P$ to $Z$ to the cabin, where should the hiker turn into the forest? How sensitive is your solution on the hiker’s pace through the woods?

![Diagram](image.png)

Figure 3.5: A hiker walks from $P$ to $Z$ to the cabin, as pictured.
Activity 3.13.

(a) Consider the region in the \(x-y\) plane that is bounded by the \(x\)-axis and the function \(f(x) = 25 - x^2\). Construct a rectangle whose base lies on the \(x\)-axis and is centered at the origin, and whose sides extend vertically until they intersect the curve \(y = 25 - x^2\). Which such rectangle has the maximum possible area? Which such rectangle has the greatest perimeter? Which has the greatest combined perimeter and area?

(b) Answer the same questions in terms of positive parameters \(a\) and \(b\) for the function \(f(x) = b - ax^2\).
Activity 3.14.

A trough is being constructed by bending a $4 \times 24$ (measured in feet) rectangular piece of sheet metal. Two symmetric folds 2 feet apart will be made parallel to the longest side of the rectangle so that the trough has cross-sections in the shape of a trapezoid, as pictured in Figure 3.6. At what angle should the folds be made to produce the trough of maximum volume? How sensitive is your solution to the 2 foot distance between the folds?

Figure 3.6: A cross-section of the trough formed by folding to an angle of $\theta$. 
Voting Questions

3.5.1 (no clicker questions for this section)
3.6 Related Rates

**Preview Activity 3.6.** A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon’s diameter is 12 inches? Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why?

(a) Draw several spheres with different radii, and observe that as volume changes, the radius, diameter, and surface area of the balloon also change.

(b) Recall that the volume of a sphere of radius $r$ is $V = \frac{4}{3}\pi r^3$. Note well that in the setting of this problem, both $V$ and $r$ are changing as time $t$ changes, and thus both $V$ and $r$ may be viewed as implicit functions of $t$, with respective derivatives $\frac{dV}{dt}$ and $\frac{dr}{dt}$.

   Differentiate both sides of the equation $V = \frac{4}{3}\pi r^3$ with respect to $t$ (using the chain rule on the right) to find a formula for $\frac{dV}{dt}$ that depends on both $r$ and $\frac{dr}{dt}$.

(c) At this point in the problem, by differentiating we have “related the rates” of change of $V$ and $r$. Recall that we are given in the problem that the balloon is being inflated at a constant rate of 20 cubic inches per second. Is this rate the value of $\frac{dr}{dt}$ or $\frac{dV}{dt}$? Why?

(d) From part (c), we know the value of $\frac{dV}{dt}$ at every value of $t$. Next, observe that when the diameter of the balloon is 12, we know the value of the radius. In the equation $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$?

(e) How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?
Activity 3.15.

A water tank has the shape of an inverted circular cone (point down) with a base of radius 6 feet and a depth of 8 feet. Suppose that water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute.

(a) Draw a picture of the conical tank, including a sketch of the water level at a point in time when the tank is not yet full. Introduce variables that measure the radius of the water’s surface and the water’s depth in the tank, and label them on your figure.

(b) Say that $r$ is the radius and $h$ the depth of the water at a given time, $t$. What equation relates the radius and height of the water, and why?

(c) Determine an equation that relates the volume of water in the tank at time $t$ to the depth $h$ of the water at that time.

(d) Through differentiation, find an equation that relates the instantaneous rate of change of water volume with respect to time to the instantaneous rate of change of water depth at time $t$.

(e) Find the instantaneous rate at which the water level is rising when the water in the tank is 3 feet deep.

(f) When is the water rising most rapidly: at $h = 3$, $h = 4$, or $h = 5$?
Activity 3.16.

A television camera is positioned 4000 feet from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. In addition, the auto-focus of the camera has to take into account the increasing distance between the camera and the rocket. We assume that the rocket rises vertically. (A similar problem is discussed and pictured dynamically at http://gvsu.edu/s/9t. Exploring the applet at the link will be helpful to you in answering the questions that follow.)

(a) Draw a figure that summarizes the given situation. What parts of the picture are changing? What parts are constant? Introduce appropriate variables to represent the quantities that are changing.

(b) Find an equation that relates the camera’s angle of elevation to the height of the rocket, and then find an equation that relates the instantaneous rate of change of the camera’s elevation angle to the instantaneous rate of change of the rocket’s height (where all rates of change are with respect to time).

(c) Find an equation that relates the distance from the camera to the rocket to the rocket’s height, as well as an equation that relates the instantaneous rate of change of distance from the camera to the rocket to the instantaneous rate of change of the rocket’s height (where all rates of change are with respect to time).

(d) Suppose that the rocket’s speed is 600 ft/sec at the instant it has risen 3000 feet. How fast is the distance from the television camera to the rocket changing at that moment? If the camera is following the rocket, how fast is the camera’s angle of elevation changing at that same moment?

(e) If from an elevation of 3000 feet onward the rocket continues to rise at 600 feet/sec, will the rate of change of distance with respect to time be greater when the elevation is 4000 feet than it was at 3000 feet, or less? Why?
Activity 3.17.

As pictured in the applet at http://gvsu.edu/s/9q, a skateboarder who is 6 feet tall rides under a 15 foot tall lamppost at a constant rate of 3 feet per second. We are interested in understanding how fast his shadow is changing at various points in time.

(a) Draw an appropriate right triangle that represents a snapshot in time of the skateboarder, lamppost, and his shadow. Let $x$ denote the horizontal distance from the base of the lamppost to the skateboarder and $s$ represent the length of his shadow. Label these quantities, as well as the skateboarder’s height and the lamppost’s height on the diagram.

(b) Observe that the skateboarder and the lamppost represent parallel line segments in the diagram, and thus similar triangles are present. Use similar triangles to establish an equation that relates $x$ and $s$.

(c) Use your work in (b) to find an equation that relates $\frac{ds}{dt}$ and $\frac{dx}{dt}$.

(d) At what rate is the length of the skateboarder’s shadow increasing at the instant the skateboarder is 8 feet from the lamppost?

(e) As the skateboarder’s distance from the lamppost increases, is his shadow’s length increasing at an increasing rate, increasing at a decreasing rate, or increasing at a constant rate?

(f) Which is moving more rapidly: the skateboarder or the tip of his shadow? Explain, and justify your answer.

\text{\textcopyright 2009--2013.
Activity 3.18.

A baseball diamond is 90′ square. A batter hits a ball along the third base line runs to first base. At what rate is the distance between the ball and first base changing when the ball is halfway to third base, if at that instant the ball is traveling 100 feet/sec? At what rate is the distance between the ball and the runner changing at the same instant, if at the same instant the runner is 1/8 of the way to first base running at 30 feet/sec?
3.6. RELATED RATES

Voting Questions

3.6.1 If \( \frac{dy}{dx} = 5 \) and \( \frac{dx}{dt} = -2 \) then \( \frac{dy}{dt} = \)
   (a) 5  (b) -2  (c) -10  (d) cannot be determined from the information given

3.6.2 If \( \frac{dz}{dx} = 12 \) and \( \frac{dy}{dx} = 2 \) then \( \frac{dz}{dy} = \)
   (a) 24  (b) 6  (c) 1/6  (d) cannot be determined from the information given

3.6.3 If \( y = 5x^2 \) and \( \frac{dx}{dt} = 3 \), then when \( x = 4 \), \( \frac{dy}{dt} = \)
   (a) 30  (b) 80  (c) 120  (d) 15x^2  (e) cannot be determined from the information given

3.6.4 The radius of a snowball changes as the snow melts. The instantaneous rate of change in radius with respect to volume is
   (a) \( \frac{dV}{dr} \)  (b) \( \frac{dr}{dV} \)  (c) \( \frac{dV}{dr} + \frac{dr}{dV} \)  (d) None of the above

3.6.5 Gravel is poured into a conical pile. The rate at which gravel is added to the pile is
   (a) \( \frac{dV}{dt} \)  (b) \( \frac{dr}{dt} \)  (c) \( \frac{dV}{dr} \)  (d) None of the above

3.6.6 Suppose a deli clerk can slice a stick of pepperoni so that its length \( L \) changes at a rate of 2 inches per minute and the total weight \( W \) of pepperoni that has been cut increases at a rate of 0.2 pounds per minute. The pepperoni weighs:
   (a) 0.4 pounds per inch  (b) 0.1 pounds per inch  (c) 10 pounds per inch  (d) 2.2 pounds per inch  (e) None of the above

3.6.7 The area of a circle, \( A = \pi r^2 \), changes as its radius changes. If the radius changes with respect to time, the change in area with respect to time is
   (a) \( \frac{dA}{dt} = 2\pi r \)  (b) \( \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \)  (c) \( \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \)  (d) Not enough information

3.6.8 As gravel is being poured into a conical pile, its volume \( V \) changes with time. As a result, the height \( h \) and radius \( r \) also change with time. Knowing that at any moment \( V = \frac{1}{3} \pi r^2 h \), the relationship between the changes in the volume, radius and height, with respect to time, is
   (a) \( \frac{dV}{dt} = \frac{1}{3} \pi \left( 2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right) \)
   (b) \( \frac{dV}{dt} = \frac{1}{3} \pi \left( 2r \frac{dr}{dt} h \right) \)
   (c) \( \frac{dV}{dt} = \frac{1}{3} \pi \left( 2r h + r^2 \frac{dh}{dt} \right) \)
   (d) \( \frac{dV}{dt} = \frac{1}{3} \pi \left( (r^2)(1) + 2r \frac{dr}{dt} h \right) \)
3.6.9 A boat is drawn close to a dock by pulling in a rope as shown. How is the rate at which the rope is pulled in related to the rate at which the boat approaches the dock?

(a) One is a constant multiple of the other.
(b) They are equal.
(c) It depends on how close the boat is to the dock.

3.6.10 A boat is drawn close to a dock by pulling in the rope at a constant rate.

The closer the boat gets to the dock, the faster it is moving.

(see figure from the previous problem)

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

3.6.11 A streetlight is mounted at the top of a pole. A man walks away from the pole. How are the rate at which he walks away from the pole and the rate at which his shadow grows related?

(a) One is a constant multiple of the other.
(b) They are equal.
(c) It depends also on how close the man is to the pole.

3.6.12 A spotlight installed in the ground shines on a wall. A woman stands between the light and the wall casting a shadow on the wall. How are the rate at which she walks away from the light and rate at which her shadow grows related?

(a) One is a constant multiple of the other.
(b) They are equal.
(c) It depends also on how close the woman is to the pole.
3.7 Global Optimization

Preview Activity 3.7. Consider the function \( f(x) = x^4 + x^3 - 7x^2 - x + 6 \).

(a) Find \( f'(x) \) and graph both \( f(x) \) and \( f'(x) \) on the same coordinate axes.

(b) Find the critical points of \( f(x) \) and use calculus to determine if these critical points are maximums or minimums on the function \( f(x) \).

(c) Now let’s create \( g(x) = f(x) + (2x + 3) \); that is, we create \( g(x) \) by adding a linear function with a slope of 2 to the function \( f(x) \).

   (i) What is the slope of the tangent line to \( g(x) \) at the critical points from \( f(x) \)? Show all of your necessary work.

   (ii) Write a few clear sentences stating why it wasn’t actually necessary to do any calculation to answer the preview question.
Activity 3.19.
Consider the equation $6x^5 - 15x^4 + 20x^3 - 30x^2 + 30x = 30$.

(a) Write the equation as $f(x) = 0$, where $f$ is a polynomial with leading term $6x^5$.

(b) Find $f'(x)$ and verify that $f'(x) = 30(x^2 + 1)(x - 1)^2$.

(c) What is the maximum number of roots that $f$ can have? What is the maximum number of solutions that the equation can have? Justify your answer using Rolle’s Theorem.
Activity 3.20.

In this activity we will prove the Mean Value Theorem based on Rolle’s Theorem. It might be worthwhile at this point to read again the comment relating both theorems which appeared just before the Mean Value Theorem was stated.

(a) Recall that the equation of a line with slope $m$ going through the point $(x_1, y_1)$ can be written using the point-slope form

$$y = m(x - x_1) + y_1.$$ 

Find the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$ and find the equation $y = s(x)$ of this secant line.

(b) The distance, to a sign, between a point $(x, f(x))$ on the graph of function $f$ and a point $(x, s(x))$ on the secant line can be obtained by taking the difference between the $y$-coordinates of the points as shown below.

Create a function $h$ which computes the distance, to a sign, between the two points.

(c) Explain why $h$ is continuous on $[a, b]$.

(d) Explain why $h$ is differentiable on $(a, b)$.

(e) Evaluate $h(a)$ and $h(b)$.

(f) Apply Rolle’s Theorem to obtain the Mean Value Theorem.

(g) The distance between Quebec City and Montreal is 233 km. A train going from Quebec City to Montreal has the following position function

$$s(t) = 233e^{-\frac{(t-3)^2}{1.5^2}}.$$
Figure 3.8: The position function $s = s(t)$ of the train.

(i) Find the average speed of the train during the first three hours.
(ii) Use a graphing device or Figure 3.8 to find approximately when the speed of the train is the same as the average speed during the first three hours.
3.7. GLOBAL OPTIMIZATION

Activity 3.21.

As mentioned, the Mean Value Theorem is mostly used to prove other results. In this activity we shall demonstrate a result that will be of use in integral calculus.

(a) Let \( x_i \) and \( x_{i+1} \) be two values on the \( x \)-axis, where \( x_i < x_{i+1} \). Suppose that \( F \) is a function that satisfies the hypotheses of the Mean Value Theorem on the interval \([x_i, x_{i+1}]\) and that the derivative of \( F \) is \( f \), that is \( F' = f \). Show that the conclusion of the Mean Value Theorem is that there exists \( x^*_i \in (x_i, x_{i+1}) \) such that

\[
F(x_{i+1}) - F(x_i) = f(x^*_i)(x_{i+1} - x_i).
\]

(b) Consider the interval \([1, 9]\).

(i) Divide the interval in 4 equal parts, that is divide the interval in 4 subintervals of equal length.

(ii) Let the 4 subintervals be \([x_0, x_1]\), \([x_1, x_2]\), \([x_2, x_3]\), and \([x_3, x_4]\). Find the values of \( x_0, x_1, x_2, x_3, \) and \( x_4 \).

(iii) We can write the difference \( F(9) - F(1) = F(x_4) - F(x_0) \) as

\[
F(9) - F(1) = F(x_4) - F(x_0),
\]

\[
= F(x_4) - F(x_3) + F(x_3) - F(x_2) + F(x_2) - F(x_1) + F(x_1) - F(x_0),
\]

\[
= [F(x_4) - F(x_3)] + [F(x_3) - F(x_2)] + [F(x_2) - F(x_1)] + [F(x_1) - F(x_0)],
\]

\[
= [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + [F(x_3) - F(x_2)] + [F(x_4) - F(x_3)].
\]

Since the subintervals have equal length, the difference between two consecutive values \( x_i \) is constant. Let the difference be \( \Delta x \), that is \( \Delta x = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3 \). Apply the Mean Value Theorem (see Equation 3.1) to each bracket in the last expression to justify the following equation

\[
F(9) - F(1) = f(x^*_1)\Delta x + f(x^*_2)\Delta x + f(x^*_3)\Delta x + f(x^*_4)\Delta x,
\]

where \( x^*_i \in (x_{i-1}, x_i) \).

(c) **(Hard)** Consider the interval \([a, b]\) and divide the interval in \( n \) subintervals of equal length \( \Delta x = (b - a)/n \) and let \( x_0, x_1, \ldots, x_{n-1}, x_n \) be the endpoints of the subintervals. Show that

\[
F(b) - F(a) = f(x^*_1)\Delta x + f(x^*_2)\Delta x + \ldots + f(x^*_n)\Delta x,
\]

where \( x^*_i \in (x_{i-1}, x_i) \). As a concluding remark, the limit

\[
\lim_{n \to \infty} f(x^*_1)\Delta x + f(x^*_2)\Delta x + \ldots + f(x^*_n)\Delta x
\]

is called a **definite integral** and it represents the net area between the function \( f \) and the \( x \)-axis for \( a \leq x \leq b \). What this exercise has shown is that this difficult limit problem can be solved by simply evaluating the net change \( F(b) - F(a) \). This result is called the **Fundamental Theorem of Calculus** and is one of the most important theorems in integral calculus/Calculus II (which you will be doing next semester), but also one of the most important theorems in mathematics in general.
Voting Questions
4.1 Determining distance traveled from velocity

Preview Activity 4.1. Suppose that a person is taking a walk along a long straight path and walks at a constant rate of 3 miles per hour.

(a) On the left-hand axes provided in Figure 4.1, sketch a labeled graph of the velocity function \( v(t) = 3 \). Note that while the scale on the two sets of axes is the same, the units on the right-hand axes differ from those on the left. The right-hand axes will be used in question (d).

![Figure 4.1: At left, axes for plotting \( y = v(t) \); at right, for plotting \( y = s(t) \).](image)

(b) How far did the person travel during the two hours? How is this distance related to the area of a certain region under the graph of \( y = v(t) \)?

(c) Find an algebraic formula, \( s(t) \), for the position of the person at time \( t \), assuming that \( s(0) = 0 \). Explain your thinking.
(d) On the right-hand axes provided in Figure 4.1, sketch a labeled graph of the position function \( y = s(t) \).

(e) For what values of \( t \) is the position function \( s \) increasing? Explain why this is the case using relevant information about the velocity function \( v \).
Activity 4.1.

Suppose that a person is walking in such a way that her velocity varies slightly according to the information given in the table below and graph given in Figure 4.2.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0.00</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
<th>1.75</th>
<th>2.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$</td>
<td>1.5000</td>
<td>1.7891</td>
<td>1.9375</td>
<td>1.9922</td>
<td>2.0000</td>
<td>2.0078</td>
<td>2.0625</td>
<td>2.2109</td>
<td>2.5000</td>
</tr>
</tbody>
</table>

![Graph of $y = v(t)$][1]

Figure 4.2: The graph of $y = v(t)$.

(a) Using the grid, graph, and given data appropriately, estimate the distance traveled by the walker during the two hour interval from $t = 0$ to $t = 2$. You should use time intervals of width $\Delta t = 0.5$, choosing a way to use the function consistently to determine the height of each rectangle in order to approximate distance traveled.

(b) How could you get a better approximation of the distance traveled on $[0, 2]$? Explain, and then find this new estimate.

(c) Now suppose that you know that $v$ is given by $v(t) = 0.5t^3 - 1.5t^2 + 1.5t + 1.5$. Remember that $v$ is the derivative of the walker’s position function, $s$. Find a formula for $s$ so that $s' = v$.

(d) Based on your work in (c), what is the value of $s(2) - s(0)$? What is the meaning of this quantity?

---

[1]: http://example.com/graph.png
Activity 4.2.

A ball is tossed vertically in such a way that its velocity function is given by \( v(t) = 32 - 32t \), where \( t \) is measured in seconds and \( v \) in feet per second. Assume that this function is valid for \( 0 \leq t \leq 2 \).

(a) For what values of \( t \) is the velocity of the ball positive? What does this tell you about the motion of the ball on this interval of time values?

(b) Find an antiderivative, \( s \), of \( v \) that satisfies \( s(0) = 0 \).

(c) Compute the value of \( s(1) - s\left(\frac{1}{2}\right) \). What is the meaning of the value you find?

(d) Using the graph of \( y = v(t) \) provided in Figure 4.3, find the exact area of the region under the velocity curve between \( t = \frac{1}{2} \) and \( t = 1 \). What is the meaning of the value you find?

(e) Answer the same questions as in (c) and (d) but instead using the interval \([0, 1]\).

(f) What is the value of \( s(2) - s(0) \)? What does this result tell you about the flight of the ball? How is this value connected to the provided graph of \( y = v(t) \)? Explain.

Figure 4.3: The graph of \( y = v(t) \).
4.1. DETERMINING DISTANCE TRAVELED FROM VELOCITY

Activity 4.3.

Suppose that an object moving along a straight line path has its velocity $v$ (in meters per second) at time $t$ (in seconds) given by the piecewise linear function whose graph is pictured in Figure 4.4. We view movement to the right as being in the positive direction (with positive velocity), while movement to the left is in the negative direction. Suppose further that the object’s initial position at time $t = 0$ is $s(0) = 1$.

(a) Determine the total distance traveled and the total change in position on the time interval $0 \leq t \leq 2$. What is the object’s position at $t = 2$?

(b) On what time intervals is the moving object’s position function increasing? Why? On what intervals is the object’s position decreasing? Why?

(c) What is the object’s position at $t = 8$? How many total meters has it traveled to get to this point (including distance in both directions)? Is this different from the object’s total change in position on $t = 0$ to $t = 8$?

(d) Find the exact position of the object at $t = 1, 2, 3, \ldots, 8$ and use this data to sketch an accurate graph of $y = s(t)$. How can you use the provided information about $y = v(t)$ to determine the concavity of $s$ on each relevant interval?
4.1. Voting Questions

4.1.1 True or False The left-sum always underestimates the area under the curve.
   
   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

4.1.2 True or False Averaging the left- and right-sums always improves your estimate.
   
   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

4.1.3 True or False When estimating an integral with right or left sums, smaller rectangles will always result in a better estimation.
   
   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

4.1.4 Consider the graph in Figure 5.1. On which interval is the left-sum approximation of the area under the curve on that interval an overestimate?

![Figure 5.1](image-url)
4.1.5 The velocities of two cars are given in Figure 5.2. Assuming that the cars start at the same place, when does Car 2 overtake Car 1?

(a) Between 0.75 and 1.25 minutes
(b) Between 1.25 and 1.75 minutes
(c) Between 1.75 and 2.25 minutes

4.1.6 You are taking a long road trip, and you look down to check your speed every 15 minutes. At 2:00 you are going 60 mph, at 2:15 you are going up a hill at 45 mph, at 2:30 you are going 65 mph, at 2:45 you are going through a canyon at 50 mph, and at 3:00 you are going 70 mph. Assume that in each 15-minute interval you are either always speeding up or always slowing down.

Using left-hand sums, how far would you estimate that you went between 2:00 and 3:00?

(a) \( \left( \frac{1}{4} \right) 60 + \left( \frac{1}{4} \right) 45 + \left( \frac{1}{4} \right) 65 + \left( \frac{1}{4} \right) 50 + \left( \frac{1}{4} \right) 70 \)
(b) \( \left( \frac{1}{4} \right) 60 + \left( \frac{1}{4} \right) 45 + \left( \frac{1}{4} \right) 65 + \left( \frac{1}{4} \right) 50 \)
(c) \( \left( \frac{1}{4} \right) 45 + \left( \frac{1}{4} \right) 65 + \left( \frac{1}{4} \right) 50 + \left( \frac{1}{4} \right) 70 \)
(d) \( \left( \frac{1}{4} \right) 60 + \left( \frac{1}{4} \right) 65 + \left( \frac{1}{4} \right) 65 + \left( \frac{1}{4} \right) 70 \)
(e) \( \left( \frac{1}{15} \right) 60 + \left( \frac{1}{15} \right) 45 + \left( \frac{1}{15} \right) 65 + \left( \frac{1}{15} \right) 50 + \left( \frac{1}{15} \right) 70 \)

4.1.7 You are taking a long road trip, and you look down to check your speed every 15 minutes. At 2:00 you are going 60 mph, at 2:15 you are going up a hill at 45 mph, at 2:30 you are going 65 mph, at 2:45 you are going through a canyon at 50 mph, and at 3:00 you are going 70 mph.
mph. Assume that in each 15-minute interval you are either always speeding up or always slowing down.

Using right-hand sums, how far would you estimate that you went between 2:00 and 3:00?

(a) \((1/4)60 + (1/4)45 + (1/4)65 + (1/4)50 + (1/4)70\)
(b) \((1/4)60 + (1/4)45 + (1/4)65 + (1/4)50\)
(c) \((1/4)45 + (1/4)65 + (1/4)50 + (1/4)70\)
(d) \((1/4)60 + (1/4)65 + (1/4)65 + (1/4)70\)

4.1.8 You are taking a long road trip, and you look down to check your speed every 15 minutes. At 2:00 you are going 60 mph, at 2:15 you are going up a hill at 45 mph, at 2:30 you are going 65 mph, at 2:45 you are going through a canyon at 50 mph, and at 3:00 you are going 70 mph. Assume that in each 15-minute interval you are either always speeding up or always slowing down.

What would be your estimate of the maximum possible distance that you could have traveled between 2:00 and 3:00?

(a) \((1/4)60 + (1/4)45 + (1/4)65 + (1/4)50 + (1/4)70\)
(b) \(1/4)60 + (1/4)45 + (1/4)65 + (1/4)50\)
(c) \((1/4)45 + (1/4)65 + (1/4)50 + (1/4)70\)
(d) \((1/4)60 + (1/4)65 + (1/4)65 + (1/4)70\)

4.1.9 The table below gives a car’s velocity, \(v\), in miles per hour, with time, \(t\), in minutes. In each of the four fifteen-minute intervals, the car is either always speeding up or always slowing down. The car’s route is a straight line with four towns on it. Town A is 60 miles from the starting point, town B is 70 miles from the starting point, town C is 73 miles from the starting point, and town D is 80 miles from the starting point.

<table>
<thead>
<tr>
<th>(t) (minutes)</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v) (miles per hour)</td>
<td>60</td>
<td>75</td>
<td>72</td>
<td>78</td>
<td>65</td>
</tr>
</tbody>
</table>

We know the car is

(a) between towns A and B.
(b) between towns B and C.
(c) between towns C and D.
(d) between towns A and D, but can’t define more clearly.
(e) past town D.
(f) None of the above.
4.2 Riemann Sums

Preview Activity 4.2. A person walking along a straight path has her velocity in miles per hour at time \( t \) given by the function \( v(t) = 0.5t^3 - 1.5t^2 + 1.5t + 1.5 \), for times in the interval \( 0 \leq t \leq 2 \). The graph of this function is also given in each of the three diagrams in Figure 4.5. Note that in each diagram, we use four rectangles to estimate the area under \( y = v(t) \) on the interval \([0, 2]\), but the method by which the four rectangles’ respective heights are decided varies among the three individual graphs.

(a) How are the heights of rectangles in the left-most diagram being chosen? Explain, and hence determine the value of

\[
S = A_1 + A_2 + A_3 + A_4
\]

by evaluating the function \( y = v(t) \) at appropriately chosen values and observing the width of each rectangle. Note, for example, that

\[
A_3 = v(1) \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1.
\]

(b) Explain how the heights of rectangles are being chosen in the middle diagram and find the value of

\[
T = B_1 + B_2 + B_3 + B_4.
\]

(c) Likewise, determine the pattern of how heights of rectangles are chosen in the right-most diagram and determine

\[
U = C_1 + C_2 + C_3 + C_4.
\]

(d) Of the estimates \( S, T, \) and \( U \), which do you think is the best approximation of \( D \), the total distance the person traveled on \([0, 2]\)? Why?
Activity 4.4.

For each sum written in sigma notation, write the sum long-hand and evaluate the sum to find its value. For each sum written in expanded form, write the sum in sigma notation.

(a) \[ \sum_{k=1}^{5} (k^2 + 2) \]

(b) \[ \sum_{i=3}^{6} (2i - 1) \]

(c) \[ 3 + 7 + 11 + 15 + \cdots + 27 \]

(d) \[ 4 + 8 + 16 + 32 \cdots + 256 \]

(e) \[ \sum_{i=1}^{6} \frac{1}{2^i} \]
Activity 4.5.

Suppose that an object moving along a straight line path has its position in feet at time $t$ in seconds given by $v(t) = \frac{2}{9}(t - 3)^2 + 2$.

(a) Carefully sketch the region whose exact area will tell you the value of the distance the object traveled on the time interval $2 \leq t \leq 5$.

(b) Estimate the distance traveled on $[2, 5]$ by computing $L_4$, $R_4$, and $M_4$.

(c) Does averaging $L_4$ and $R_4$ result in the same value as $M_4$? If not, what do you think the average of $L_4$ and $R_4$ measures?

(d) For this question, think about an arbitrary function $f$, rather than the particular function $v$ given above. If $f$ is positive and increasing on $[a, b]$, will $L_n$ over-estimate or under-estimate the exact area under $f$ on $[a, b]$? Will $R_n$ over- or under-estimate the exact area under $f$ on $[a, b]$? Explain.
Activity 4.6.

Suppose that an object moving along a straight line path has its velocity $v$ (in feet per second) at time $t$ (in seconds) given by

$$v(t) = \frac{1}{2}t^2 - 3t + \frac{7}{2}.$$

(a) Compute $M_5$, the middle Riemann sum, for $v$ on the time interval $[1, 5]$. Be sure to clearly identify the value of $\Delta t$ as well as the locations of $t_0, t_1, \cdots, t_5$. In addition, provide a careful sketch of the function and the corresponding rectangles that are being used in the sum.

(b) Building on your work in (a), estimate the total change in position of the object on the interval $[1, 5]$.

(c) Building on your work in (a) and (b), estimate the total distance traveled by the object on $[1, 5]$.

(d) Use appropriate computing technology\(^1\) to compute $M_{10}$ and $M_{20}$. What exact value do you think the middle sum eventually approaches as $n$ increases without bound? What does that number represent in the physical context of the overall problem?

\(^1\)For instance, consider the applet at http://gvsu.edu/s/a9 and change the function and adjust the locations of the blue points that represent the interval endpoints $a$ and $b$.\)
Voting Questions

4.2.1 (none)


4.3 The Definite Integral

**Preview Activity 4.3.** Consider the applet found at [http://gvsu.edu/s/aw](http://gvsu.edu/s/aw). There, you will initially see the situation shown in Figure 4.6. Observe that we can change the window in which

![Image of a left Riemann sum](image_url)

Figure 4.6: A left Riemann sum with 5 subintervals for the function \( f(x) = \frac{3}{1 + x^2} \) on the interval \([-5, 5]\). The value of the sum is \( L_5 = 7.43076923 \).

the function is viewed, as well as the function itself. Set the minimum and maximum values of \( x \) and \( y \) so that we view the function on the window where \( 1 \leq x \leq 4 \) and \(-1 \leq y \leq 12\), where the function is \( f(x) = 2x + 1 \) (note that you need to enter “\(2\times x+1\)” as the function’s formula). You should see the updated figure shown in Figure 4.7. Note that the value of the Riemann sum of our choice is displayed in the upper left corner of the window. Further, by updating the value in the “Intervals” window and/or the “Method”, we can see the different value of the Riemann sum that arises by clicking the “Compute!” button.

(a) Update the applet so that the function being considered is \( f(x) = 2x + 1 \) on \([1, 4]\), as directed above. For this function on this interval, compute \( L_n, M_n, R_n \) for \( n = 10, n = 100, \) and \( n = 1000 \). What do you conjecture is the exact area bounded by \( f(x) = 2x + 1 \) and the \( x \)-axis on \([1, 4]\)?

(b) Use basic geometry to determine the exact area bounded by \( f(x) = 2x + 1 \) and the \( x \)-axis on \([1, 4]\).

(c) Based on your work in (a) and (b), what do you observe occurs when we increase the number of subintervals used in the Riemann sum?

(d) Update the applet to consider the function \( f(x) = x^2 + 1 \) on the interval \([1, 4]\) (note that you will want to increase the maximum value of \( y \) to at least 17, and you need to enter “\(x^2 + 1\)” as the function’s formula). The exact area bounded by \( f(x) = x^2 + 1 \) and the \( x \)-axis on \([1, 4]\) is \( 17 \frac{1}{3} \).
4.3. THE DEFINITE INTEGRAL

Figure 4.7: A left Riemann sum with 5 subintervals for the function \( f(x) = 2x + 1 \) on the interval \([1, 4]\). The value of the sum is \( L_5 = 16.2 \).

1” for the function formula. Use the applet to compute \( L_n, M_n, R_n \) for \( n = 10, n = 100, \) and \( n = 1000 \). What do you conjecture is the exact area bounded by \( f(x) = x^2 + 1 \) and the \( x \)-axis on \([1, 4]\)?

(e) Why can we not compute the exact value of the area bounded by \( f(x) = x^2 + 1 \) and the \( x \)-axis on \([1, 4]\) using a formula like we did in (b)?

\( \square \)
Activity 4.7.

Use known geometric formulas and the net signed area interpretation of the definite integral to evaluate each of the definite integrals below.

(a) \( \int_{0}^{1} 3x \, dx \)

(b) \( \int_{-1}^{4} (2 - 2x) \, dx \)

(c) \( \int_{-1}^{1} \sqrt{1 - x^2} \, dx \)

(d) \( \int_{-3}^{4} g(x) \, dx \), where \( g \) is the function pictured in Figure 4.8. Assume that each portion of \( g \) is either part of a line or part of a circle.

Figure 4.8: A function \( g \) that is piecewise defined; each piece of the function is part of a circle or part of a line.
Activity 4.8.

Suppose that the following information is known about the functions $f$, $g$, $x^2$, and $x^3$:

- $\int_2^0 f(x) \, dx = -3$; $\int_5^2 f(x) \, dx = 2$
- $\int_2^0 g(x) \, dx = 4$; $\int_2^5 g(x) \, dx = -1$
- $\int_2^0 x^2 \, dx = \frac{8}{3}$; $\int_5^2 f(x) \, dx = \frac{117}{3}$
- $\int_2^0 x^3 \, dx = 4$; $\int_2^5 x^3 \, dx = \frac{609}{4}$

Use the provided information and the rules discussed in the preceding section to evaluate each of the following definite integrals.

(a) $\int_5^2 f(x) \, dx$

(b) $\int_0^5 g(x) \, dx$

(c) $\int_0^5 (f(x) + g(x)) \, dx$

(d) $\int_5^2 (3x^2 - 4x^3) \, dx$

(e) $\int_0^5 (2x^3 - 7g(x)) \, dx$
Suppose that \( v(t) = \sqrt{4 - (t - 2)^2} \) tells us the instantaneous velocity of a moving object on the interval \( 0 \leq t \leq 4 \), where \( t \) is measured in minutes and \( v \) is measured in meters per minute.

(a) Sketch an accurate graph of \( y = v(t) \). What kind of curve is \( y = \sqrt{4 - (t - 2)^2} \)?

(b) Evaluate \( \int_0^4 v(t) \, dt \) exactly.

(c) In terms of the physical problem of the moving object with velocity \( v(t) \), what is the meaning of \( \int_0^4 v(t) \, dt \)? Include units on your answer.

(d) Determine the exact average value of \( v(t) \) on \([0, 4]\). Include units on your answer.

(e) Sketch a rectangle whose base is the line segment from \( t = 0 \) to \( t = 4 \) on the \( t \)-axis such that the rectangle’s area is equal to the value of \( \int_0^4 v(t) \, dt \). What is the rectangle’s exact height?

(f) How can you use the average value you found in (d) to compute the total distance traveled by the moving object over \([0, 4]\)?
Voting Questions

4.3.1 Which of the following is the best estimate of \( \int_{0}^{3} f(x) \, dx \), where \( f(x) \) is given in the figure below?

- (a) 13
- (b) 17
- (c) 65
- (d) 85

4.3.2 Which of the following is the best estimate of \( \int_{-2}^{2} f(x) \, dx \), where \( f(x) \) is given in the figure below?

- (a) -4
- (b) -6
- (c) 3
- (d) 6
- (e) 12

4.3.3 Make a sketch of the function \( f(x) = \cos x \) and decide whether \( \int_{0}^{\pi} f(x) \, dx \) is:

- (a) Positive
- (b) Negative
- (c) Zero

4.3.4 Make a sketch of the function \( f(x) = -x^3 \) and decide whether \( \int_{-5}^{5} f(x) \, dx \) is:
4.3. THE DEFINITE INTEGRAL

(a) Positive
(b) Negative
(c) Zero

4.3.5 True or False: If a piece of string has been chopped into \( n \) small pieces and the \( i^{th} \) piece is \( \Delta x_i \) inches long, then the total length of the string is exactly \( \sum_{i=1}^{n} \Delta x_i \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

4.3.6 You want to estimate the area underneath the graph of a positive function by using four rectangles of equal width. The rectangles that must give the best estimate of this area are those with height obtained from the:

(a) Left endpoints
(b) Midpoints
(c) Right endpoints
(d) Not enough information

4.3.7 Suppose you are slicing an 11-inch long carrot REALLY thin from the greens end to the tip of the root. If each slice has a circular cross section \( f(x) = \pi [r(x)]^2 \) for each \( x \) between 0 and 11, and we make our cuts at \( x_1, x_2, x_3, \ldots, x_n \) then a good approximation for the volume of the carrot is

(a) \( \sum_{i=1}^{n} f(x_i)x_i \)
(b) \( \sum_{i=1}^{n} [f(x_{i+1} - f(x_i))]x_i \)
(c) \( \sum_{i=1}^{n} f(x_i)[x_{i+1} - x_i] \)
(d) None of the above.

4.3.8 Let \( f \) be a continuous function on the interval \([a, b]\).

True or False: \( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i \) may lead to different limits if we choose the \( x_i^* \) to be the left-endpoints instead of midpoints.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
Preview Activity 4.4. A student with a third floor dormitory window 32 feet off the ground tosses a water balloon straight up in the air with an initial velocity of 16 feet per second. It turns out that the instantaneous velocity of the water balloon is given by the velocity function \( v(t) = -32t + 16 \), where \( v \) is measured in feet per second and \( t \) is measured in seconds.

(a) Let \( s(t) \) represent the height of the water balloon above the ground at time \( t \), and note that \( s \) is an antiderivative of \( v \). That is, \( v \) is the derivative of \( s \): \( s'(t) = v(t) \). Find a formula for \( s(t) \) that satisfies the initial condition that the balloon is tossed from 32 feet above ground. In other words, make your formula for \( s \) satisfy \( s(0) = 32 \).

(b) At what time does the water balloon reach its maximum height? At what time does the water balloon land?

(c) Compute the three differences \( s(\frac{1}{2}) - s(0) \), \( s(2) - s(\frac{1}{2}) \), and \( s(2) - s(0) \). What do these differences represent?

(d) What is the total vertical distance traveled by the water balloon from the time it is tossed until the time it lands?

(e) Sketch a graph of the velocity function \( y = v(t) \) on the time interval \([0, 2]\). What is the total net signed area bounded by \( y = v(t) \) and the \( t \)-axis on \([0, 2]\)? Answer this question in two ways: first by using your work above, and then by using a familiar geometric formula to compute areas of certain relevant regions.
Activity 4.10.

Use the Fundamental Theorem of Calculus to evaluate each of the following integrals exactly. For each, sketch a graph of the integrand on the relevant interval and write one sentence that explains the meaning of the value of the integral in terms of the (net signed) area bounded by the curve.

(a) \( \int_{-1}^{4} (2 - 2x) \, dx \)

(b) \( \int_{0}^{\pi/2} \sin(x) \, dx \)

(c) \( \int_{0}^{1} e^{x} \, dx \)

(d) \( \int_{-1}^{1} x^{5} \, dx \)

(e) \( \int_{0}^{2} (3x^{3} - 2x^{2} - e^{x}) \, dx \)
270 4.4. THE FUNDAMENTAL THEOREM OF CALCULUS & BASIC ANTIDERIVATIVES

<table>
<thead>
<tr>
<th>given function, $f(x)$</th>
<th>antiderivative, $F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k, (k \neq 0)$</td>
<td></td>
</tr>
<tr>
<td>$x^n, n \neq -1$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{x}, x &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td></td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td></td>
</tr>
<tr>
<td>$\sec(x) \tan(x)$</td>
<td></td>
</tr>
<tr>
<td>$\csc(x) \cot(x)$</td>
<td></td>
</tr>
<tr>
<td>$\sec^2(x)$</td>
<td></td>
</tr>
<tr>
<td>$\csc^2(x)$</td>
<td></td>
</tr>
<tr>
<td>$e^x$</td>
<td></td>
</tr>
<tr>
<td>$a^x \ (a &gt; 1)$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{1+x^2}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: Familiar basic functions and their antiderivatives.

**Activity 4.11.**

Use your knowledge of derivatives of basic functions to complete the above table of antiderivatives. For each entry, your task is to find a function $F$ whose derivative is the given function $f$. When finished, use the FTC and the results in the table to evaluate the three given definite integrals.

(a) \( \int_0^1 (x^3 - x - e^x + 2) \, dx \)

(b) \( \int_0^{\pi/3} (2 \sin(t) - 4 \cos(t) + \sec^2(t) - \pi) \, dt \)

(c) \( \int_0^1 (\sqrt{x} - x^2) \, dx \)
Activity 4.12.

During a 30-minute workout, a person riding an exercise machine burns calories at a rate of $c$ calories per minute, where the function $y = c(t)$ is given in Figure 4.9. On the interval $0 \leq t \leq 10$, the formula for $c$ is $c(t) = -0.05t^2 + t + 10$, while on $20 \leq t \leq 30$, its formula is $c(t) = -0.05t^2 + 2t - 5$.

Figure 4.9: The rate $c(t)$ at which a person exercising burns calories, measured in calories per minute.

(a) What is the exact total number of calories the person burns during the first 10 minutes of her workout?

(b) Let $C(t)$ be an antiderivative of $c(t)$. What is the meaning of $C(30) - C(0)$ in the context of the person exercising? Include units on your answer.

(c) Determine the exact average rate at which the person burned calories during the 30-minute workout.

(d) At what time(s), if any, is the instantaneous rate at which the person is burning calories equal to the average rate at which she burns calories, on the time interval $0 \leq t \leq 30$?
Voting Questions

4.4.1 On what interval is the average value of $\sin x$ the smallest?

(a) $0 \leq x \leq \frac{\pi}{2}$
(b) $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$
(c) $0 \leq x \leq \pi$
(d) $0 \leq x \leq \frac{3\pi}{2}$

4.4.2 Water is pouring out of a pipe at the rate of $f(t)$ gallons/minute. You collect the water that flows from the pipe between $t = 2$ minutes and $t = 4$ minutes. The amount of water you collect can be represented by:

(a) $\int_{2}^{4} f(t)\,dt$
(b) $f(4) - f(2)$
(c) $(4 - 2)f(4)$
(d) the average of $f(4)$ and $f(2)$ times the amount of time that elapsed

4.4.3 If $f(t)$ is measured in gallons/minute and $t$ is measured in minutes, what are the units of $\int_{2}^{4} f(t)\,dt$?

(a) gallons/minute
(b) gallons
(c) minutes
(d) gallons/minute/minute

4.4.4 A bicyclist starts from home and rides back and forth along a straight east/west highway. Her velocity is as shown in the graph, where positive velocities indicate travel toward the east, and negative velocities indicate travel toward the west. On what time intervals is she stopped?
4.4.5 A bicyclist starts from home and rides back and forth along a straight east/west highway. Her velocity is as shown in the graph, where positive velocities indicate travel toward the east, and negative velocities indicate travel toward the west. How far from home is she the first time she stops, and in what direction?

(a) 3 feet east
(b) 3 feet west
(c) 60 feet east
(d) 60 feet west
(e) 90 feet east
(f) 90 feet west
(g) 3600 feet east
(h) 3600 feet west

4.4.6 A bicyclist starts from home and rides back and forth along a straight east/west highway. Her velocity is as shown in the graph, where positive velocities indicate travel toward the east, and negative velocities indicate travel toward the west. At what time does she bike past her house?
4.4.7 A bicyclist starts from home and rides back and forth along a straight east/west highway. Her velocity as a function of time is given by \( v(t) \). What does \( \int_{0}^{11} v(t) \, dt \) represent?

(a) The total distance the bicyclist rode in eleven minutes
(b) The bicyclist’s average velocity over eleven minutes
(c) The bicyclist’s distance from the home after eleven minutes
(d) None of the above

4.4.8 A bicyclist starts from home and rides back and forth along a straight east/west highway. Her velocity as a function of time is given by \( v(t) \). What does \( \int_{0}^{11} |v(t)| \, dt \) represent?

(a) The total distance the bicyclist rode in eleven minutes
(b) The bicyclist’s average velocity over eleven minutes
(c) The bicyclist’s distance from the home after eleven minutes
(d) None of the above.

4.4.9 A bicyclist starts from home and rides back and forth along a straight east/west highway. Her velocity as a function of time is given by \( v(t) \). What does \( \frac{1}{11} \int_{0}^{11} v(t) \, dt \) represent?

(a) The total distance the bicyclist rode in eleven minutes
(b) The bicyclists average velocity over eleven minutes
(c) The bicyclist’s distance from the home after eleven minutes
(d) None of the above.
4.4.10 The graph shows the derivative of a function $f$. If $f(0) = 3$, what is $f(2)$?

(a) 2
(b) 4
(c) 7
(d) None of the above

4.4.11 The graph shows the derivative of a function $f$. Which is greater?

(a) $f(2) - f(0)$
(b) $f(3) - f(1)$
(c) $f(4) - f(2)$

4.4.12 Suppose $f$ is a differentiable function. Then $\int_0^5 f'(t)dt = f(5)$

(a) Always
(b) Sometimes
(c) Never

4.4.13 If $f$ is continuous and $f(x) < 0$ for all $x$ in $[a, b]$, then $\int_a^b f(x)dx$

(a) must be negative
(b) might be 0
4.4.14 Let $f$ be a continuous function on the interval $[a, b]$. **True or False:** There exist two constants $m$ and $M$, such that

$$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a).$$

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident  

4.4.15 You are traveling with velocity $v(t)$ that varies continuously over the interval $[a, b]$ and your position at time $t$ is given by $s(t)$. Which of the following represent your average velocity for that time interval:

I. $\frac{\int_a^b v(t)dt}{b - a}$

II. $\frac{s(b) - s(a)}{b - a}$

III. $v(c)$ for at least one $c$ between $a$ and $b$

(a) I, II, and III  
(b) I only  
(c) I and II only  
(d) II only  
(e) II and III only  

4.4.16 **True or False:** $\int_0^2 f(x)dx = \int_0^2 f(t)dt$

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident
4.4.17 **True or False:** If \( a = b \) then \( \int_a^b f(x)\,dx = 0 \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

4.4.18 **True or False:** If \( a \neq b \) then \( \int_a^b f(x)\,dx \neq 0 \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

4.4.19 **True or False:** If \( a \neq b \) and if \( \int_a^b f(x)\,dx = 0 \), then \( f(x) = 0 \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

4.4.20 **True or False:** If \( a \neq b \) and if \( \int_a^b |f(x)|\,dx = 0 \), then \( f(x) = 0 \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

4.4.21 **True or False:** If \( \int_0^2 f(x)\,dx = 3 \) and \( \int_2^4 f(x)\,dx = 5 \), then \( \int_0^4 f(x)\,dx = 8 \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident
4.4.22 Given that \( \int_{0}^{2} f(x) \, dx = 3 \) and \( \int_{2}^{4} f(x) \, dx = 5 \), what is \( \int_{0}^{2} f(2x) \, dx \)?

(a) \( 3/2 \)
(b) 3
(c) 4
(d) 6
(e) 8
(f) Cannot be determined

4.4.23 **True or False:** If \( \int_{0}^{2} (f(x) + g(x)) \, dx = 10 \) and \( \int_{0}^{2} f(x) \, dx = 3 \), then \( \int_{0}^{2} g(x) \, dx = 7 \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

4.4.24 **True or False:** \( \int_{1}^{2} f(x) \, dx + \int_{2}^{3} g(x) \, dx = \int_{1}^{3} (f(x) + g(x)) \, dx \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

4.4.25 **True or False:** If \( f(x) \leq g(x) \) for \( 2 \leq x \leq 6 \), then \( \int_{2}^{6} f(x) \, dx \leq \int_{2}^{6} g(x) \, dx \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

4.4.26 **True or False:** If \( \int_{2}^{6} f(x) \, dx \leq \int_{2}^{6} g(x) \, dx \), then \( f(x) \leq g(x) \) for \( 2 \leq x \leq 6 \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
4.4.27 **True or False:** If \( f \) is continuous on the interval \([a, b]\), then \( \int_a^b f(x) \, dx \) is a number (rather than a function).

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident

4.4.28 \( \int (x^3 + 5) \, dx = \)

(a) \( 3x^2 \)  
(b) \( 3x^2 + 5 \)  
(c) \( \frac{x^4}{4} + 5 \)  
(d) \( \frac{1}{4}x^4 + 5x \)  
(e) None of the above

4.4.29 \( \int \sin x \, dx = \)

(a) \( \sin x + C \)  
(b) \( \cos x + C \)  
(c) \( -\sin x + C \)  
(d) \( -\cos x + C \)  
(e) None of the above

4.4.30 \( \int x\sin x \, dx = \)

(a) \( \cos x + C \)  
(b) \( \frac{1}{2}x^2(\cos x) + C \)  
(c) \( x\cos x + C \)  
(d) \( \frac{1}{2}x^2\sin x + C \)  
(e) Cant do with what we know right now

4.4.31 \( \int 5e^x \, dx = \)
4.4. THE FUNDAMENTAL THEOREM OF CALCULUS & BASIC ANTIDERIVATIVES

(a) \(5e^x + C\)
(b) \(e^x + C\)
(c) \(5xe^x + C\)
(d) \(\frac{5e^{x+1}}{x+1} + C\)
(e) None of the above

4.4.32 \(\int \sqrt{x} \, dx = \)

(a) \(\frac{1}{2}x^{-1/2} + C\)
(b) \(\frac{2}{5}x^{3/2} + C\)
(c) \(\frac{3}{2}x^{3/2} + C\)
(d) \(\frac{3}{2}x^{2/3} + C\)
(e) Can’t do with what we know right now

4.4.33 \(\int \sqrt{x^7} \, dx = \)

(a) \(x^{3/2} + C\)
(b) \(\frac{5}{2}x^{5/2} + C\)
(c) \(\frac{3}{2}x^{1/2} + C\)
(d) \(\frac{2}{5}x^{5/2} + C\)
(e) \(\frac{2}{3}x^{5/3} + C\)
(f) None of the above

4.4.34 \(\int \frac{7}{x^7} \, dx = \)

(a) \(-\frac{7}{4}x^{-4} + C\)
(b) \(7x^{-5} + C\)
(c) \(-\frac{7}{6x^6} + C\)
(d) \(\frac{7}{4x^4} + C\)
(e) None of the above

4.4.35 What is \(\int_1^5 3 \, dt?\)

(a) 3
4.4. THE FUNDAMENTAL THEOREM OF CALCULUS & BASIC ANTIDERIVATIVES

(b) 4
(c) 12
(d) 15
(e) 16

4.4.36 What is $\int \frac{5}{x^2} \, dx$?
(a) $-\frac{5}{x} + C$
(b) $\frac{5}{x^2} + C$
(c) $-\frac{10}{x^3} + C$
(d) $\frac{30}{x^4} + C$

4.4.37 $\int \frac{3}{x} \, dx =$
(a) $-\frac{3}{2}x^{-2} + C$
(b) $3 \ln x + C$
(c) $\frac{3}{x^2} + C$
(d) $3x^{-1} + C$
(e) None of the above

4.4.38 An antiderivative of $6x^2$ is
(a) $2x^3$
(b) $2x^3 + 5$
(c) $2x^3 + 18$
(d) $2x^3 - 6$
(e) All of the above

4.4.39 Which of the following is an antiderivative of $y(x) = 3 \sin(x) + 7$?
(a) $g(x) = 3 \cos(x)$
(b) $g(x) = 3 \cos(x) + 7$
(c) $g(x) = 3 \cos(x) + 7x$
(d) $g(x) = -3 \cos(x) + 7x$
4.4.40 **True or False:** If $F(x)$ is an antiderivative of $f(x)$ and $G(x) = F(x) + 2$, then $G(x)$ is an antiderivative of $f(x)$.

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident

4.4.41 Water is flowing out of a reservoir at a rate given by $f(t) = 5000 + 50t + 5t^2$, where $t$ is in days and $f$ is in gallons per day. How much water flows out of the reservoir during the first week?

(a) 572 gallons  
(b) 5,000 gallons  
(c) 5,595 gallons  
(d) 35,000 gallons  
(e) 36,797 gallons

4.4.42 Trucks are driving over a bridge at a rate given by the function $b(t) = 30 \cos(t) + 70$, where $t$ is in hours from noon and $b$ is in trucks per hour. How many trucks drive across the bridge between 3pm and 6pm?

(a) 13 trucks  
(b) 210 trucks  
(c) 197 trucks  
(d) 269 trucks
Chapter 5

Finding Antiderivatives and Evaluating Integrals

5.1 Constructing Accurate Graphs of Antiderivatives

Preview Activity 5.1. Suppose that the following information is known about a function $f$: the graph of its derivative, $y = f'(x)$, is given in Figure 5.1. Further, assume that $f'$ is piecewise linear (as pictured) and that for $x \leq 0$ and $x \geq 6$, $f'(x) = 0$. Finally, it is given that $f(0) = 1$.

![Figure 5.1: At left, the graph of $y = f'(x)$; at right, axes for plotting $y = f(x)$.](image)

(a) On what interval(s) is $f$ an increasing function? On what intervals is $f$ decreasing?

(b) On what interval(s) is $f$ concave up? concave down?

(c) At what point(s) does $f$ have a relative minimum? a relative maximum?
(d) Recall that the Total Change Theorem tells us that

\[ f(1) - f(0) = \int_0^1 f'(x) \, dx. \]

What is the exact value of \( f(1) \)?

(e) Use the given information and similar reasoning to that in (d) to determine the exact value of \( f(2), f(3), f(4), f(5), \) and \( f(6) \).

(f) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of \( y = f(x) \) on the axes provided, being sure to indicate the behavior of \( f \) for \( x < 0 \) and \( x > 6 \).
Activity 5.1.

Suppose that the function \( y = f(x) \) is given by the graph shown in Figure 5.2, and that the pieces of \( f \) are either portions of lines or portions of circles. In addition, let \( F \) be an antiderivative of \( f \) and say that \( F(0) = -1 \). Finally, assume that for \( x \leq 0 \) and \( x \geq 7 \), \( f(x) = 0 \).

![Graph of \( y = f(x) \)](image)

**Figure 5.2: At left, the graph of \( y = f(x) \).**

(a) On what interval(s) is \( F \) an increasing function? On what intervals is \( F \) decreasing?

(b) On what interval(s) is \( F \) concave up? concave down? neither?

(c) At what point(s) does \( F \) have a relative minimum? a relative maximum?

(d) Use the given information to determine the exact value of \( F(x) \) for \( x = 1, 2, \ldots, 7 \). In addition, what are the values of \( F(-1) \) and \( F(8) \)?

(e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of \( y = F(x) \) on the axes provided, being sure to indicate the behavior of \( F \) for \( x < 0 \) and \( x > 7 \). Clearly indicate the scale on the vertical and horizontal axes of your graph.

(f) What happens if we change one key piece of information: in particular, say that \( G \) is an antiderivative of \( f \) and \( G(0) = 0 \). How (if at all) would your answers to the preceding questions change? Sketch a graph of \( G \) on the same axes as the graph of \( F \) you constructed in (e).

\<
Activity 5.2.

For each of the following functions, sketch an accurate graph of the antiderivative that satisfies the given initial condition. In addition, sketch the graph of two additional antiderivatives of the given function, and state the corresponding initial conditions that each of them satisfy. If possible, find an algebraic formula for the antiderivative that satisfies the initial condition.

(a) original function: \( g(x) = |x| - 1 \);
   initial condition: \( G(-1) = 0 \);
   interval for sketch: \([-2, 2]\)

(b) original function: \( h(x) = \sin(x) \);
   initial condition: \( H(0) = 1 \);
   interval for sketch: \([0, 4\pi]\)

(c) original function: \( p(x) = \begin{cases} 
  x^2, & \text{if } 0 < x \leq 1 \\
  -(x - 2)^2, & \text{if } 1 < x < 2 \\
  0 & \text{otherwise}
\end{cases} \)
   initial condition: \( P(0) = 1 \);
   interval for sketch: \([-1, 3]\)
Activity 5.3.

Suppose that $g$ is given by the graph at left in Figure 5.3 and that $A$ is the corresponding integral function defined by $A(x) = \int_1^x g(t) \, dt$.

![Graph of $y = g(t)$](image)

![Axes for plotting $y = A(x)$](image)

Figure 5.3: At left, the graph of $y = g(t)$; at right, axes for plotting $y = A(x)$, where $A$ is defined by the formula $A(x) = \int_1^x g(t) \, dt$.

(a) On what interval(s) is $A$ an increasing function? On what intervals is $A$ decreasing? Why?

(b) On what interval(s) do you think $A$ is concave up? concave down? Why?

(c) At what point(s) does $A$ have a relative minimum? a relative maximum?

(d) Use the given information to determine the exact values of $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, $A(5)$, and $A(6)$.

(e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = A(x)$ on the axes provided, being sure to indicate the behavior of $A$ for $x < 0$ and $x > 6$.

(f) How does the graph of $B$ compare to $A$ if $B$ is instead defined by $B(x) = \int_0^x g(t) \, dt$?
Voting Questions

5.1.1 Which of the graphs (a-d) could represent an antiderivative of the function shown in Figure 6.1.

5.1.2 Which of the graphs (a-d) could represent an antiderivative of the function shown in Figure 6.2.

5.1.3 Consider the graph of $f'(x)$ shown below. Which of the functions with values from the table could represent $f(x)$?
5.1. CONSTRUCTING ACCURATE GRAPHS OF ANTIDERIVATIVES

Table 5.1

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>y(x)</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(b)</td>
<td>b(x)</td>
<td>5</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>(c)</td>
<td>f(x)</td>
<td>32</td>
<td>34</td>
<td>35</td>
</tr>
<tr>
<td>(d)</td>
<td>k(x)</td>
<td>10</td>
<td>12</td>
<td>14</td>
</tr>
</tbody>
</table>

1. (a) only
2. (a), (b), and (c) only
3. All of them
4. None of them

5.1.4 Figure 6.4 shows $f'(x)$. If $f(2) = 5$, what is $f(0)$?

(a) 0
(b) 3
(c) 7
(d) Can’t tell

5.1.5 The graph of $f$ is given below. Let $F'(x) = f(x)$. Where does $F$ have critical points?
5.1.6 The graph of $f$ is given below. Let $F'(x) = f(x)$. Where does $F$ have a global max on $[0, 2\pi]$?

(a) $x = 0, \pi, 2\pi$

(b) $x = \pi$

(c) $x = \pi/2, 3\pi/2$

(d) None of the above
5.1.7 The derivative, $f'$, of a function $f$ is plotted below. At approximately what value of $x$ does $f$ reach a maximum, on the range $[0, 10]$?

(a) $x = 0$
(b) $x = \pi/2$
(c) $x = \pi$
(d) $x = 3\pi/2$
(e) $x = 2\pi$
5.1.8 The derivative, $f'$, of a function $f$ is plotted below. If we know that the maximum value of $f$ on this range is 20, what is $f(9.5)$?

(a) 1  
(b) 2.5  
(c) 4  
(d) 7  
(e) 9.5
5.1.9 The derivative, $f'$, of a function $f$ is plotted below. When is $f$ concave up?

(a) $f(9.5) \approx 6$
(b) $f(9.5) \approx 14$
(c) $f(9.5) \approx -14$
(d) $f(9.5) \approx 34$
(a) $x > 5$

(b) $x < 5$

(c) $x < 2.5$ and $x > 7.5$

(d) $2.5 < x < 7.5$

(e) $1 < x < 4$ and $x > 9.5$

5.1.10 The graph below shows the second derivative, $f''$ of a function, and we know $f(1) = 3$ and $f'(1) = 0$. Is $f'(2)$ positive or negative?
5.1. CONSTRUCTING ACCURATE GRAPHS OF ANTIDERIVATIVES

5.1.11 The graph below shows the second derivative, \( f'' \) of a function, and we know \( f(1) = 3 \) and \( f'(1) = 0 \). Is \( f(-3) \) bigger than 3 or smaller than 3?

(a) \( f''(2) > 0 \)
(b) \( f''(2) < 0 \)
(c) It is impossible to tell without further information.

5.1.11 The graph below shows the second derivative, \( f'' \) of a function, and we know \( f(1) = 3 \) and \( f'(1) = 0 \). Is \( f(-3) \) bigger than 3 or smaller than 3?

(a) \( f(-3) > 3 \)
5.1. CONSTRUCTING ACCURATE GRAPHS OF ANTIDERIVATIVES

(b) \( f(-3) < 3 \)
(c) It is impossible to tell without further information.

5.1.12 The figure below is the graph of \( f'(x) \). Where is the global maximum of \( f \) on \([-4, 4]\)?

(a) \( x = -3.2 \)
(b) \( x = -2 \)
(c) \( x = -0.8 \)
(d) \( x = 2 \)
(e) \( x = 4 \)
5.2 The Second Fundamental Theorem of Calculus

Preview Activity 5.2. Consider the function \( A \) defined by the rule
\[
A(x) = \int_1^x f(t) \, dt,
\]
where \( f(t) = 4 - 2t \).

(a) Compute \( A(1) \) and \( A(2) \) exactly.

(b) Use the First Fundamental Theorem of Calculus to find an equivalent formula for \( A(x) \) that does not involve integrals. That is, use the first FTC to evaluate \( \int_1^x (4 - 2t) \, dt \).

(c) Observe that \( f \) is a linear function; what kind of function is \( A \)?

(d) Using the formula you found in (b) that does not involve integrals, compute \( A'(x) \).

(e) While we have defined \( f \) by the rule \( f(t) = 4 - 2t \), it is equivalent to say that \( f \) is given by the rule \( f(x) = 4 - 2x \). What do you observe about the relationship between \( A \) and \( f \)?
Activity 5.4.

Suppose that \( f \) is the function given in Figure 5.4 and that \( f \) is a piecewise function whose parts are either portions of lines or portions of circles, as pictured. In addition, let \( A \) be the function defined by the rule \( A(x) = \int_{2}^{x} f(t) \, dt \).

(a) What does the Second FTC tell us about the relationship between \( A \) and \( f \)?

(b) Compute \( A(1) \) and \( A(3) \) exactly.

(c) Sketch a precise graph of \( y = A(x) \) on the axes at right that accurately reflects where \( A \) is increasing and decreasing, where \( A \) is concave up and concave down, and the exact values of \( A \) at \( x = 0, 1, \ldots, 7 \).

(d) How is \( A \) similar to, but different from, the function \( F \) that you found in Activity 5.1?

(e) With as little additional work as possible, sketch precise graphs of the functions \( B(x) = \int_{3}^{x} f(t) \, dt \) and \( C(x) = \int_{1}^{x} f(t) \, dt \). Justify your results with at least one sentence of explanation.

\( \triangleright \)
Activity 5.5.
Suppose that \( f(t) = \frac{t}{1+t^2} \) and \( F(x) = \int_0^x f(t) \, dt \).

(a) On the axes at left in Figure 5.5, plot a graph of \( f(t) = \frac{t}{1+t^2} \) on the interval \( -10 \leq t \leq 10 \). Clearly label the vertical axes with appropriate scale.

(b) What is the key relationship between \( F \) and \( f \), according to the Second FTC?

(c) Use the first derivative test to determine the intervals on which \( F \) is increasing and decreasing.

(d) Use the second derivative test to determine the intervals on which \( F \) is concave up and concave down. Note that \( f'(t) \) can be simplified to be written in the form \( f'(t) = \frac{1-t^2}{(1+t^2)^2} \).

(e) Using technology appropriately, estimate the values of \( F(5) \) and \( F(10) \) through appropriate Riemann sums.

(f) Sketch an accurate graph of \( y = F(x) \) on the righthand axes provided, and clearly label the vertical axes with appropriate scale.
Activity 5.6.

Evaluate each of the following derivatives and definite integrals. Clearly cite whether you use the First or Second FTC in so doing.

(a) \( \frac{d}{dx} \left[ \int_{4}^{x} e^{t^2} \, dt \right] \)

(b) \( \int_{-2}^{x} \frac{d}{dt} \left[ \frac{t^4}{1 + t^4} \right] \, dt \)

(c) \( \frac{d}{dx} \left[ \int_{x}^{1} \cos(t^3) \, dt \right] \)

(d) \( \int_{3}^{x} \frac{d}{dt} \left[ \ln(1 + t^2) \right] \, dt \)

(e) \( \frac{d}{dx} \left[ \int_{1}^{x^3} \sin(t^2) \, dt \right] \)

(Hint: Let \( F(x) = \int_{4}^{x} \sin(t^2) \, dt \) and observe that this problem is asking you to evaluate \( \frac{d}{dx} [F(x^3)] \).
5.2. THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

Voting Questions

5.2.1 If \( f(x) = \int_1^x t^4 \, dt \), then

(a) \( f'(x) = t^4 \)

(b) \( f'(x) = x^4 \)

(c) \( f'(x) = \frac{1}{5}x^5 - \frac{1}{5} \)

(d) \( f'(x) = x^4 - 1 \)

5.2.2 If \( f(t) = \int_t^7 \cos x \, dx \), then

(a) \( f'(t) = \cos t \)

(b) \( f'(t) = \sin t \)

(c) \( f'(t) = \sin 7 - \sin t \)

(d) \( f'(t) = -\cos t \)

(e) \( f'(t) = -\sin t \)

5.2.3 If \( f(x) = \int_2^x e^{2t} \, dt \), then

(a) \( f'(x) = 2xe^{2x^2} \)

(b) \( f'(x) = e^{2x} \)

(c) \( f'(x) = e^{2x^2} \)

(d) \( f'(x) = 2e^{2x^2} \)

(e) \( f'(x) = \frac{1}{2}e^{2x^2} - \frac{1}{2}e^8 \)

5.2.4 True or False: If \( f \) is continuous on the interval \([a, b]\), then \( \frac{d}{dx} \int_a^b f(x) \, dx = f(x) \).

(a) True, and I am very confident

(b) True, but I am not very confident

(c) False, but I am not very confident

(d) False, and I am very confident

5.2.5 If \( f \) is continuous on the interval \([a, b]\), then \( \frac{d}{dx} \int_a^b f(x) \, dx = \)

(a) 0
5.2. THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

(b) \( f(b) \)
(c) \( f(x) \)
(d) None of the above.

5.2.6 **True or False:** \( \int_0^x \sin(t^2) \, dt \) is an antiderivative of \( \sin(x^2) \).

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

5.2.7 The graph of function \( f \) is given below. Let \( g(x) = \int_0^x f(t) \, dt \). Then for \( 0 < x < 2 \), \( g(x) \) is

(a) increasing and concave up.
(b) increasing and concave down.
(c) decreasing and concave up.
(d) decreasing and concave down.

5.2.8 The graph of function \( f \) is given below. Let \( g(x) = \int_0^x f(t) \, dt \). Then
5.2. THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

(a) \( g(0) = 0, g'(0) = 0 \) and \( g'(2) = 0 \)
(b) \( g(0) = 0, g'(0) = 4 \) and \( g'(2) = 0 \)
(c) \( g(0) = 1, g'(0) = 0 \) and \( g'(2) = 1 \)
(d) \( g(0) = 0, g'(0) = 0 \) and \( g'(2) = 1 \)

5.2.9 The speed of a car is given by the function \( s(t) = 15t^2 \), where \( t \) is in seconds, and \( s \) is in feet per second. If the car starts out a distance of 20 ft from the starting line, how far from the starting line will the car be at \( t = 4 \) seconds?

(a) 240 ft
(b) 260 ft
(c) 320 ft
(d) 340 ft
(e) 6,000 ft

5.2.10 The function \( g(x) \) is related to the function \( f(x) \) by the equation \( g(x) = \int_{\frac{x}{3}}^{x} f(x)dx \), and \( g(x) \) is plotted below. Where is \( f(x) \) positive?
(a) $3 < x < 8$
(b) $x < 6$
(c) $2.5 < x$
(d) $x < 2.5$
5.3 Integration by Substitution

**Preview Activity 5.3.** In Section 2.5, we learned the Chain Rule and how it can be applied to find the derivative of a composite function. In particular, if \( u \) is a differentiable function of \( x \), and \( f \) is a differentiable function of \( u(x) \), then

\[
\frac{d}{dx} [f(u(x))] = f'(u(x)) \cdot u'(x).
\]

In words, we say that the derivative of a composite function \( c(x) = f(u(x)) \), where \( f \) is considered the “outer” function and \( u \) the “inner” function, is “the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function.”

(a) For each of the following functions, use the Chain Rule to find the function’s derivative. Be sure to label each derivative by name (e.g., the derivative of \( g(x) \) should be labeled \( g'(x) \)).

i. \( g(x) = e^{3x} \)

ii. \( h(x) = \sin(5x + 1) \)

iii. \( p(x) = \arctan(2x) \)

iv. \( q(x) = (2 - 7x)^4 \)

v. \( r(x) = 3^{4-11x} \)

(b) For each of the following functions, use your work in (a) to help you determine the general antiderivative\(^1\) of the function. Label each antiderivative by name (e.g., the antiderivative of \( m \) should be called \( M \)). In addition, check your work by computing the derivative of each proposed antiderivative.

i. \( m(x) = e^{3x} \)

ii. \( n(x) = \cos(5x + 1) \)

iii. \( s(x) = \frac{1}{1+4x^2} \)

iv. \( v(x) = (2 - 7x)^3 \)

v. \( w(x) = 3^{4-11x} \)

(c) Based on your experience in parts (a) and (b), conjecture an antiderivative for each of the following functions. Test your conjectures by computing the derivative of each proposed antiderivative.

---

\(^1\)Recall that the general antiderivative of a function includes “+C” to reflect the entire family of functions that share the same derivative.
i. $a(x) = \cos(\pi x)$

ii. $b(x) = (4x + 7)^{11}$

iii. $c(x) = xe^{x^2}$
Activity 5.7.

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

(a) \( \int \sin(8 - 3x) \, dx \)

(b) \( \int \sec^2(4x) \, dx \)

(c) \( \int \frac{1}{11x-9} \, dx \)

(d) \( \int \csc(2x + 1) \cot(2x + 1) \, dx \)

(e) \( \int \frac{1}{\sqrt{1-16x^2}} \, dx \)

(f) \( \int 5^x \, dx \)
Activity 5.8.

Evaluate each of the following indefinite integrals by using these steps:

- Find two functions within the integrand that form (up to a possible missing constant) a function-derivative pair;
- Make a substitution and convert the integral to one involving $u$ and $du$;
- Evaluate the new integral in $u$;
- Convert the resulting function of $u$ back to a function of $x$ by using your earlier substitution;
- Check your work by differentiating the function of $x$. You should come up with the integrand originally given.

(a) \[ \int \frac{x^2}{5x^3 + 1} \, dx \]

(b) \[ \int e^x \sin(e^x) \, dx \]

(c) \[ \int \frac{\cos(\sqrt{x})}{\sqrt{x}} \, dx \]
Activity 5.9.

Evaluate each of the following definite integrals exactly through an appropriate $u$-substitution.

(a) $\int_{1}^{2} \frac{x}{1 + 4x^2} \, dx$

(b) $\int_{0}^{1} e^{-x}(2e^{-x} + 3)^9 \, dx$

(c) $\int_{\pi/2}^{4\pi} \frac{\cos \left( \frac{1}{x} \right)}{x^2} \, dx$
5.3. INTEGRATION BY SUBSTITUTION

Voting Questions

5.3.1 True or False: \( \int 2xe^{x^2} \, dx = e^{x^2} + C. \)

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

5.3.2 If we are trying to evaluate the integral \( \int e^{\cos \theta} \sin \theta \, d\theta \), which substitution would be most helpful?

(a) \( u = \cos \theta \)
(b) \( u = \sin \theta \)
(c) \( u = e^{\cos \theta} \)

5.3.3 If we are trying to evaluate the integral \( \int x^2 \sqrt{x^3 + 5} \, dx \), which substitution would be most helpful?

(a) \( u = x^2 \)
(b) \( u = x^3 \)
(c) \( u = x^3 + 5 \)
(d) \( u = \sqrt{x^3 + 5} \)

5.3.4 Would a substitution be useful in evaluating this integral? \( \int x \sin(x^2) \, dx \)

(a) Yes, I think substitution would be useful, and I am very confident.
(b) Yes, I think substitution would be useful, but I am not very confident.
(c) No, I think substitution would not be useful, but I am not very confident.
(d) No, I think substitution would not be useful, and I am very confident.

5.3.5 Would a substitution be useful in evaluating this integral? \( \int x \sin x \, dx \)

(a) Yes, I think substitution would be useful, and I am very confident.
(b) Yes, I think substitution would be useful, but I am not very confident.
(c) No, I think substitution would not be useful, but I am not very confident.
5.3. INTEGRATION BY SUBSTITUTION

(d) No, I think substitution would not be useful, and I am very confident.

5.3.6 Would a substitution be useful in evaluating this integral? \( \int (3x + 2)(x^3 + 5x)^7 \, dx \)

(a) Yes, I think substitution would be useful, and I am very confident.
(b) Yes, I think substitution would be useful, but I am not very confident.
(c) No, I think substitution would not be useful, but I am not very confident.
(d) No, I think substitution would not be useful, and I am very confident.

5.3.7 Would a substitution be useful in evaluating this integral? \( \int \frac{1}{\ln x} \, dx \)

(a) Yes, I think substitution would be useful, and I am very confident.
(b) Yes, I think substitution would be useful, but I am not very confident.
(c) No, I think substitution would not be useful, but I am not very confident.
(d) No, I think substitution would not be useful, and I am very confident.

5.3.8 Would a substitution be useful in evaluating this integral? \( \int e^{\sin \theta} \cos \theta \, d\theta \)

(a) Yes, I think substitution would be useful, and I am very confident.
(b) Yes, I think substitution would be useful, but I am not very confident.
(c) No, I think substitution would not be useful, but I am not very confident.
(d) No, I think substitution would not be useful, and I am very confident.

5.3.9 Would a substitution be useful in evaluating this integral? \( \int e^x \sqrt{1 + e^x} \, dx \)

(a) Yes, I think substitution would be useful, and I am very confident.
(b) Yes, I think substitution would be useful, but I am not very confident.
(c) No, I think substitution would not be useful, but I am not very confident.
(d) No, I think substitution would not be useful, and I am very confident.

5.3.10 Would a substitution be useful in evaluating this integral? \( \int \frac{\sin x}{x} \, dx \)

(a) Yes, I think substitution would be useful, and I am very confident.
(b) Yes, I think substitution would be useful, but I am not very confident.
(c) No, I think substitution would not be useful, but I am not very confident.
312 5.3. INTEGRATION BY SUBSTITUTION

(d) No, I think substitution would not be useful, and I am very confident.

5.3.11 Would a substitution be useful in evaluating this integral? \( \int \frac{e^x - e^{-x}}{(e^x + e^{-x})^3} \, dx \)

(a) Yes, I think substitution would be useful, and I am very confident.
(b) Yes, I think substitution would be useful, but I am not very confident.
(c) No, I think substitution would not be useful, but I am not very confident.
(d) No, I think substitution would not be useful, and I am very confident.

5.3.12 Would a substitution be useful in evaluating this integral? \( \int x^{16}(x^{17} + 16x)^{16} \, dx \)

(a) Yes, I think substitution would be useful, and I am very confident.
(b) Yes, I think substitution would be useful, but I am not very confident.
(c) No, I think substitution would not be useful, but I am not very confident.
(d) No, I think substitution would not be useful, and I am very confident.

5.3.13 What is \( \int_{0}^{1/2} \cos(\pi x) \, dx \)?

(a) 0
(b) \( \pi \)
(c) \( 1/\pi \)
(d) 1
(e) This integral cannot be done with substitution.

5.3.14 A company’s sales are growing at an exponential rate, so that the sales rate is \( R = R_0 e^{0.2t} \) widgets per year, where \( t \) is in years, starting now. Right now the company is selling widgets at a rate of 1000 widgets per year. If this model holds, how many widgets will they sell over the next ten years?

(a) 1,278
(b) 6,389
(c) 7,389
(d) 31,945
(e) 32,945
(f) 36,945
5.3.15 What is \( \int \frac{1}{\sqrt{4-x}} \, dx \)?

(a) \( \frac{1}{2} (4-x)^{-3/2} + C \)
(b) \( 2\sqrt{4-x} + C \)
(c) \( -2\sqrt{4-x} + C \)
(d) \( -\frac{2}{3} (4-x)^{3/2} + C \)
(e) This integral cannot be done with substitution.

5.3.16 What is \( \int \frac{1}{5x} \, dx \)?

(a) \( \ln(5x) + C \)
(b) \( \frac{1}{5} \ln x + C \)
(c) \( \frac{1}{5} \ln(5x) + C \)
(d) This integral cannot be done with substitution.

5.3.17 What is \( \int xe^{x^2} \, dx \)?

(a) \( \frac{1}{2} e^u + C \)
(b) \( -\frac{1}{2} e^{-x^2} + C \)
(c) \( -2e^{-x^2} + C \)
(d) \( e^{-x^2} - 4x^2 e^{-x^2} + C \)
(e) This integral cannot be done with substitution.

5.3.18 What is \( \int \cos x \sin^6 x \, dx \)?

(a) \( \frac{1}{7} x^7 + C \)
(b) \( \frac{1}{7} \sin^7 x + C \)
(c) \( \frac{1}{7} \cos^7 x + C \)
(d) This integral cannot be done with substitution.

5.3.19 What is \( \int \cos x \sin x \, dx \)?

(a) \( \frac{1}{7} \sin^2 x + C \)
(b) \( -\frac{1}{2} \cos^2 x + C \)
(c) \( \frac{1}{2} \sin^2 x \cos^2 x + C \)
(d) This integral cannot be done with substitution.

5.3.20 What is \( \int \frac{\sin x}{\cos x} \, dx \)?

(a) \(- \ln(\cos x) + C\)
(b) \(\ln(\sin x) + C\)
(c) \(- \ln \left( \frac{\sin x}{\cos x} \right) + C\)
(d) \(\ln(\cos x) + C\)
(e) This integral cannot be done with substitution.

5.3.21 \( \int \tan x \, dx = \)

(a) \(\sec^2 x + C\)
(b) \(\ln |\cos x| + C\)
(c) \(\ln |\sec x| + C\)
(d) \(\ln |\sec x + \tan x| + C\)

5.3.22 \( \int \tan^2 x \, dx = \)

(a) \(\tan x - x + C\)
(b) \(\sec^2 x + C\)
(c) \(\sec x \tan x + C\)
(d) \(\sec x + C\)
5.4 Integration by Parts

Preview Activity 5.4. In Section 2.3, we developed the Product Rule and studied how it is employed to differentiate a product of two functions. In particular, recall that if $f$ and $g$ are differentiable functions of $x$, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

(a) For each of the following functions, use the Product Rule to find the function’s derivative. Be sure to label each derivative by name (e.g., the derivative of $g(x)$ should be labeled $g'(x)$).

i. $g(x) = x \sin(x)$

ii. $h(x) = xe^x$

iii. $p(x) = x \ln(x)$

iv. $q(x) = x^2 \cos(x)$

v. $r(x) = e^x \sin(x)$

(b) Use your work in (a) to help you evaluate the following indefinite integrals. Use differentiation to check your work.

i. $\int xe^x + e^x \, dx$

ii. $\int e^x(\sin(x) + \cos(x)) \, dx$

iii. $\int 2x \cos(x) - x^2 \sin(x) \, dx$

iv. $\int x \cos(x) + \sin(x) \, dx$

v. $\int 1 + \ln(x) \, dx$

(c) Observe that the examples in (b) work nicely because of the derivatives you were asked to calculate in (a). Each integrand in (b) is precisely the result of differentiating one of the products of basic functions found in (a). To see what happens when an integrand is still a product but not necessarily the result of differentiating an elementary product, we consider how to evaluate

$$\int x \cos(x) \, dx.$$
i. First, observe that

\[
\frac{d}{dx}[x \sin(x)] = x \cos(x) + \sin(x).
\]

Integrating both sides indefinitely and using the fact that the integral of a sum is the sum of the integrals, we find that

\[
\int \left( \frac{d}{dx}[x \sin(x)] \right) \, dx = \int x \cos(x) \, dx + \int \sin(x) \, dx.
\]

In this last equation, evaluate the indefinite integral on the left side as well as the rightmost indefinite integral on the right.

ii. In the most recent equation from (i.), solve the equation for the expression \( \int x \cos(x) \, dx \).

iii. For which product of basic functions have you now found the antiderivative?
Activity 5.10.

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

(a) \( \int t e^{-t} \, dt \)

(b) \( \int 4x \sin(3x) \, dx \)

(c) \( \int z \sec^2(z) \, dz \)

(d) \( \int x \ln(x) \, dx \)

\( \triangleleft \)
Activity 5.11.

Evaluate each of the following indefinite integrals, using the provided hints.

(a) Evaluate $\int \arctan(x) \, dx$ by using Integration by Parts with the substitution $u = \arctan(x)$ and $dv = 1 \, dx$.

(b) Evaluate $\int \ln(z) \, dz$. Consider a similar substitution to the one in (a).

(c) Use the substitution $z = t^2$ to transform the integral $\int t^3 \sin(t^2) \, dt$ to a new integral in the variable $z$, and evaluate that new integral by parts.

(d) Evaluate $\int s^5 e^{s^3} \, ds$ using an approach similar to that described in (c).

(e) Evaluate $\int e^{2t} \cos(e^t) \, dt$. You will find it helpful to note that $e^{2t} = e^t \cdot e^t$.

\"
Activity 5.12.
Evaluate each of the following indefinite integrals.

(a) \( \int x^2 \sin(x) \, dx \)

(b) \( \int t^3 \ln(t) \, dt \)

(c) \( \int e^z \sin(z) \, dz \)

(d) \( \int s^2 e^{3s} \, ds \)

(e) \( \int t \arctan(t) \, dt \)

\textbf{(Hint:} At a certain point in this problem, it is very helpful to note that \( \frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2} \).) \textbf{\triangledown}
Voting Questions

5.4.1 What is the derivative of \( f(x) = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + 25? \)

(a) \( f'(x) = xe^{3x} \)
(b) \( f'(x) = \frac{2}{3}e^{3x} \)
(c) \( f'(x) = \frac{1}{3}e^{3x} + xe^{3x} \)
(d) \( f'(x) = e^{3x} \)

5.4.2 What is \( \int xe^{4x} \, dx \)?

(a) \( \frac{1}{5}x^2e^{4x} + C \)
(b) \( \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C \)
(c) \( \frac{1}{4}xe^{4x} - \frac{1}{4}e^{4x} + C \)
(d) \( \frac{1}{16}e^{4x} - \frac{1}{4}xe^{4x} + C \)

5.4.3 Find \( \int_1^4 \ln(t)\sqrt{t} \, dt \).

(a) 4.28
(b) 3.83
(c) -1
(d) 0.444
(e) 5.33
(f) This integral cannot be done with integration by parts.

5.4.4 Estimate \( \int_0^5 f(x)g'(x) \, dx \) if \( f(x) = 2x \) and \( g(x) \) is given in the figure below.
5.4. INTEGRATION BY PARTS

(a) 40
(b) 20
(c) 10
(d) −10
(e) This integral cannot be done with integration by parts.

5.4.5 Find an antiderivative of \( x^2e^x \).

(a) \( x^2e^x - 2xe^x + 2e^x \)
(b) \( x^2e^x - 2xe^x \)
(c) \( \frac{1}{3}x^3e^x - x^2e^x + 2e^x \)
(d) \( x^2e^x - 2xe^x - 2e^x \)
(e) This integral cannot be done with integration by parts.

5.4.6 How many applications of integration by parts are required to evaluate \( \int x^3e^x \, dx \)?

(a) 1
(b) 2
(c) 3
(d) The integral cannot be evaluated using integration by parts.

5.4.7 \( \int x \cos 2x \, dx = \)

(a) \( x \sin 2x + \frac{1}{2} \cos 2x \)
(b) \( \frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x \)
(c) \( \frac{1}{2}x \sin 2x - \frac{1}{4} \cos 2x \)
(d) None of the above
5.5 Other Options for Finding Algebraic Antiderivatives

Preview Activity 5.5. For each of the indefinite integrals below, the main question is to decide whether the integral can be evaluated using \( u \)-substitution, integration by parts, a combination of the two, or neither. For integrals for which your answer is affirmative, state the substitution(s) you would use. It is not necessary to actually evaluate any of the integrals completely, unless the integral can be evaluated immediately using a familiar basic antiderivative.

(a) \( \int x^2 \sin(x^3) \, dx \), \( \int x^2 \sin(x) \, dx \), \( \int \sin(x^3) \, dx \), \( \int x^5 \sin(x^3) \, dx \)

(b) \( \int \frac{1}{1 + x^2} \, dx \), \( \int \frac{x}{1 + x^2} \, dx \), \( \int \frac{2x + 3}{1 + x^2} \, dx \), \( \int \frac{e^x}{1 + (e^x)^2} \, dx \),

(c) \( \int x \ln(x) \, dx \), \( \int \frac{\ln(x)}{x} \, dx \), \( \int \ln(1 + x^2) \, dx \), \( \int x \ln(1 + x^2) \, dx \),

(d) \( \int x \sqrt{1 - x^2} \, dx \), \( \int \frac{1}{\sqrt{1 - x^2}} \, dx \), \( \int \frac{x}{\sqrt{1 - x^2}} \, dx \), \( \int \frac{1}{x \sqrt{1 - x^2}} \, dx \)
Activity 5.13.

Activity. For each of the following integrals, evaluate the integral by using the partial fraction decomposition provided.

(a) \[ \int \frac{1}{x^2 - 2x - 3} \, dx, \quad \text{given that} \quad \frac{1}{x^2 - 2x - 3} = \frac{1}{4} \frac{1}{x - 3} - \frac{1}{4} \frac{1}{x + 1} \]

(b) \[ \int \frac{x^2 + 1}{x^3 - x^2} \, dx, \quad \text{given that} \quad \frac{x^2 + 1}{x^3 - x^2} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x - 1} \]

(c) \[ \int \frac{x - 2}{x^4 + x^2} \, dx, \quad \text{given that} \quad \frac{x - 2}{x^4 + x^2} = \frac{1}{x} - \frac{2}{x^2} + \frac{-x + 2}{1 + x^2} \]
Activity 5.14.

Activity. For each of the following integrals, evaluate the integral using $u$-substitution and/or an entry from the table found in Appendix ??.

(a) $\int \sqrt{x^2 + 4} \, dx$

(b) $\int x \sqrt{x^2 + 4} \, dx$

(c) $\int \frac{2}{\sqrt{16 + 25x^2}} \, dx$

(d) $\int \frac{1}{x^2\sqrt{49 - 36x^2}} \, dx$
Voting Questions

5.5.1 What trigonometric substitution should be made to evaluate \( \int \frac{dx}{(16 - x^2)^{3/2}} \)?

(a) \( u = 16 - x^2 \)
(b) \( x = 16 \sin \theta \)
(c) \( x = 4 \sin \theta \)
(d) This integral cannot be evaluated using trigonometric substitution.

5.5.2 In determining the partial fraction decomposition of \( \frac{x^2 + 1}{x^4 - 4x^3 - 32x^2} \), how many coefficients are there to solve for?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5
(f) 6
5.6 Numerical Integration

### Preview Activity 5.6

As we begin to investigate ways to approximate definite integrals, it will be insightful to compare results to integrals whose exact values we know. To that end, the following sequence of questions centers on $\int_{0}^{3} x^2 \, dx$.

(a) Use the applet\(^2\) at [http://gvsu.edu/s/dP](http://gvsu.edu/s/dP) with the function $f(x) = x^2$ on the window of $x$ values from 0 to 3 and $y$ values from $-1$ to 10, to compute $L_3$, the left Riemann sum with three subintervals.

(b) Likewise, use the applet to compute $R_3$ and $M_3$, the right and middle Riemann sums with three subintervals, respectively.

(c) Use the Fundamental Theorem of Calculus to compute the exact value of $I = \int_{0}^{3} x^2 \, dx$.

(d) We define the **error** in an approximation of a definite integral to be the difference between the integral’s exact value and the approximation’s value. What is the error that results from using $L_3$? From $R_3$? From $M_3$?

(e) In what follows in this section, we will learn a new approach to estimating the value of a definite integral known as the Trapezoid Rule. For now, use the “Trapezoid” option in the applet in the pull-down menu for the “Method” of estimating the definite integral, and determine the approximation generated by 3 trapezoids. What is the error in this approximation? How does it compare to the errors you calculated in (d)?

(f) What is the formula for the area of a trapezoid with bases of length $b_1$ and $b_2$ and height $h$?

---

\(^2\)Mike May, St. Louis University, [http://gvsu.edu/s/dQ](http://gvsu.edu/s/dQ).
Activity 5.15.

In this activity, we explore the relationships among the errors generated by left, right, midpoint, and trapezoid approximations to the definite integral $\int_1^2 \frac{1}{x^2} \, dx$

(a) Use the First FTC to evaluate $\int_1^2 \frac{1}{x^2} \, dx$ exactly.

(b) Use appropriate computing technology to compute the following approximations for $\int_1^2 \frac{1}{x^2} \, dx$: $T_4$, $M_4$, $T_8$, and $M_8$.

(c) Let the error of an approximation be the difference between the exact value of the definite integral and the resulting approximation. For instance, if we let $E_{T,4}$ represent the error that results from using the trapezoid rule with 4 subintervals to estimate the integral, we have

$$E_{T,4} = \int_1^2 \frac{1}{x^2} \, dx - T_4.$$  

Similarly, we compute the error of the midpoint rule approximation with 8 subintervals by the formula

$$E_{M,8} = \int_1^2 \frac{1}{x^2} \, dx - M_8.$$  

Based on your work in (a) and (b) above, compute $E_{T,4}$, $E_{T,8}$, $E_{M,4}$, $E_{M,8}$.

(d) Which rule consistently over-estimates the exact value of the definite integral? Which rule consistently under-estimates the definite integral?

(e) What behavior(s) of the function $f(x) = \frac{1}{x^2}$ lead to your observations in (d)?

<
Activity 5.16.

A car traveling along a straight road is braking and its velocity is measured at several different points in time, as given in the following table. Assume that \( v \) is continuous, always decreasing, and always decreasing at a decreasing rate, as is suggested by the data.

<table>
<thead>
<tr>
<th>seconds, ( t )</th>
<th>0</th>
<th>0.3</th>
<th>0.6</th>
<th>0.9</th>
<th>1.2</th>
<th>1.5</th>
<th>1.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity in ft/sec, ( v(t) )</td>
<td>100</td>
<td>99</td>
<td>96</td>
<td>90</td>
<td>80</td>
<td>50</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Plot the given data on the set of axes provided in Figure 5.6 with time on the horizontal axis and the velocity on the vertical axis.

(b) What definite integral will give you the exact distance the car traveled on \([0, 1.8]\)?

(c) Estimate the total distance traveled on \([0, 1.8]\) by computing \( L_3 \), \( R_3 \), and \( T_3 \). Which of these under-estimates the true distance traveled?

(d) Estimate the total distance traveled on \([0, 1.8]\) by computing \( M_3 \). Is this an over- or under-estimate? Why?

(e) Use your results from (c) and (d) improve your estimate further by using Simpson’s Rule.

(f) What is your best estimate of the average velocity of the car on \([0, 1.8]\)? Why? What are the units on this quantity?

\[
\begin{array}{c}
\text{\( v \)} \\
\end{array}
\]
\[
\begin{array}{c}
\text{\( t \)} \\
\end{array}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5_6}
\caption{Axes for plotting the data in Activity 5.16.}
\end{figure}
Activity 5.17.
Consider the functions \( f(x) = 2 - x^2 \), \( g(x) = 2 - x^3 \), and \( h(x) = 2 - x^4 \), all on the interval \([0, 1]\).
For each of the questions that require a numerical answer in what follows, write your answer exactly in fraction form.

(a) On the three sets of axes provided in Figure 5.7, sketch a graph of each function on the interval \([0, 1]\), and compute \( L_1 \) and \( R_1 \) for each. What do you observe?

(b) Compute \( M_1 \) for each function to approximate \( \int_0^1 f(x) \, dx \), \( \int_0^1 g(x) \, dx \), and \( \int_0^1 h(x) \, dx \), respectively.

(c) Compute \( T_1 \) for each of the three functions, and hence compute \( S_1 \) for each of the three functions.

(d) Evaluate each of the integrals \( \int_0^1 f(x) \, dx \), \( \int_0^1 g(x) \, dx \), and \( \int_0^1 h(x) \, dx \) exactly using the First FTC.

(e) For each of the three functions \( f \), \( g \), and \( h \), compare the results of \( L_1 \), \( R_1 \), \( M_1 \), \( T_1 \), and \( S_1 \) to the true value of the corresponding definite integral. What patterns do you observe?

Figure 5.7: Axes for plotting the functions in Activity 5.17.
Voting Questions

5.6.1
Chapter 6

Using Definite Integrals

6.1 Using Definite Integrals to Find Area and Length

Preview Activity 6.1. Consider the functions given by $f(x) = 5 - (x - 1)^2$ and $g(x) = 4 - x$.

(a) Use algebra to find the points where the graphs of $f$ and $g$ intersect.

(b) Sketch an accurate graph of $f$ and $g$ on the axes provided, labeling the curves by name and the intersection points with ordered pairs.

(c) Find and evaluate exactly an integral expression that represents the area between $y = f(x)$ and the $x$-axis on the interval between the intersection points of $f$ and $g$.

(d) Find and evaluate exactly an integral expression that represents the area between $y = g(x)$ and the $x$-axis on the interval between the intersection points of $f$ and $g$.

(e) What is the exact area between $f$ and $g$ between their intersection points? Why?
Activity 6.1.

In each of the following problems, our goal is to determine the area of the region described. For each region, (i) determine the intersection points of the curves, (ii) sketch the region whose area is being found, (iii) draw and label a representative slice, and (iv) state the area of the representative slice. Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region’s area.

(a) The finite region bounded by \( y = \sqrt{x} \) and \( y = \frac{1}{4} x \).

(b) The finite region bounded by \( y = 12 - 2x^2 \) and \( y = x^2 - 8 \).

(c) The area bounded by the y-axis, \( f(x) = \cos(x) \), and \( g(x) = \sin(x) \), where we consider the region formed by the first positive value of \( x \) for which \( f \) and \( g \) intersect.

(d) The finite regions between the curves \( y = x^3 - x \) and \( y = x^2 \).
Activity 6.2.

In each of the following problems, our goal is to determine the area of the region described. For each region, (i) determine the intersection points of the curves, (ii) sketch the region whose area is being found, (iii) draw and label a representative slice, and (iv) state the area of the representative slice. Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region’s area. **Note well:** At the step where you draw a representative slice, you need to make a choice about whether to slice vertically or horizontally.

(a) The finite region bounded by $x = y^2$ and $x = 6 - 2y^2$.
(b) The finite region bounded by $x = 1 - y^2$ and $x = 2 - 2y^2$.
(c) The area bounded by the $x$-axis, $y = x^2$, and $y = 2 - x$.
(d) The finite regions between the curves $x = y^2 - 2y$ and $y = x$. 

\[\Box\]
Activity 6.3.

Each of the following questions somehow involves the arc length along a curve.

(a) Use the definition and appropriate computational technology to determine the arc length along \( y = x^2 \) from \( x = -1 \) to \( x = 1 \).

(b) Find the arc length of \( y = \sqrt{4 - x^2} \) on the interval \( 0 \leq x \leq 4 \). Find this value in two different ways: (a) by using a definite integral, and (b) by using a familiar property of the curve.

(c) Determine the arc length of \( y = xe^{3x} \) on the interval \([0, 1]\).

(d) Will the integrals that arise calculating arc length typically be ones that we can evaluate exactly using the First FTC, or ones that we need to approximate? Why?

(e) A moving particle is traveling along the curve given by \( y = f(x) = 0.1x^2 + 1 \), and does so at a constant rate of 7 cm/sec, where both \( x \) and \( y \) are measured in cm (that is, the curve \( y = f(x) \) is the path along which the object actually travels; the curve is not a “position function”). Find the position of the particle when \( t = 4 \) sec, assuming that when \( t = 0 \), the particle’s location is \((0, f(0))\).
6.1. USING DEFINITE INTEGRALS TO FIND AREA AND LENGTH

Voting Questions

6.1.1 Find the area of the region \( R \) bounded by the graphs of \( f(x) = x^2 \) and \( g(x) = \sqrt{x} \).

(a) \( \frac{1}{3} \)
(b) \( \frac{7}{6} \)
(c) \( -\frac{1}{3} \)
(d) None of the above

6.1.2 The length of the graph of \( y = \sin(x^2) \) from \( x = 0 \) to \( x = 2\pi \) is calculated by

(a) \( \int_{0}^{2\pi} (1 + \sin(x^2)) \, dx \)
(b) \( \int_{0}^{2\pi} \sqrt{1 + \sin(x^2)} \, dx \)
(c) \( \int_{0}^{2\pi} \sqrt{1 + (2x \sin(x^2))^2} \, dx \)
(d) \( \int_{0}^{2\pi} \sqrt{1 + (2x \cos(x^2))^2} \, dx \)
6.2 Using Definite Integrals to Find Volume

Preview Activity 6.2. Consider a circular cone of radius 3 and height 5, which we view horizontally as pictured in Figure 6.2. Our goal in this activity is to use a definite integral to determine the volume of the cone.

(a) Find a formula for the linear function $y = f(x)$ that is pictured in Figure 6.2.

(b) For the representative slice of thickness $\Delta x$ that is located horizontally at a location $x$ (somewhere between $x = 0$ and $x = 5$), what is the radius of the representative slice? Note that the radius depends on the value of $x$.

(c) What is the volume of the representative slice you found in (b)?

(d) What definite integral will sum the volumes of the thin slices across the full horizontal span of the cone? What is the exact value of this definite integral?

(e) Compare the result of your work in (d) to the volume of the cone that comes from using the formula $V_{\text{cone}} = \frac{1}{3} \pi r^2 h$.

Figure 6.2: The circular cone described in Preview Activity 6.2
Activity 6.4.

In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find.

(a) The region $S$ bounded by the $x$-axis, the curve $y = \sqrt{x}$, and the line $x = 4$; revolve $S$ about the $x$-axis.

(b) The region $S$ bounded by the $y$-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve $S$ about the $x$-axis.

(c) The finite region $S$ bounded by the curves $y = \sqrt{x}$ and $y = x^3$; revolve $S$ about the $x$-axis.

(d) The finite region $S$ bounded by the curves $y = 2x^2 + 1$ and $y = x^2 + 4$; revolve $S$ about the $x$-axis.

(e) The region $S$ bounded by the $y$-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve $S$ about the $y$-axis. How does the problem change considerably when we revolve about the $y$-axis?
Activity 6.5.

In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find.

(a) The region $S$ bounded by the $y$-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve $S$ about the $y$-axis.

(b) The region $S$ bounded by the $x$-axis, the curve $y = \sqrt{x}$, and the line $x = 4$; revolve $S$ about the $y$-axis.

(c) The finite region $S$ in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$; revolve $S$ about the $x$-axis.

(d) The finite region $S$ in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$; revolve $S$ about the $y$-axis.

(e) The finite region $S$ bounded by the curves $x = (y - 1)^2$ and $y = x - 1$; revolve $S$ about the $y$-axis.
Activity 6.6.

In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find. For each prompt, use the finite region $S$ in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$.

(a) Revolve $S$ about the line $y = -2$.

(b) Revolve $S$ about the line $y = 4$.

(c) Revolve $S$ about the line $x = -1$.

(d) Revolve $S$ about the line $x = 5$. 

\end{document}
Voting Questions

6.2.1 If we slice a cone with a circular base parallel to the $x$-axis, the resulting slices would look like

(a) Circles  
(b) Triangles  
(c) Cylinders with a circular base  
(d) Cylinders with a triangular base  
(e) Cones

6.2.2 If we slice a cone with a circular base parallel to the $x$-axis, then the thickness of the slices is given by

(a) $\Delta x$  
(b) $\Delta y$  
(c) $x$  
(d) $y$

6.2.3 If we put the tip of a cone with a circular base at the origin and let it open upward, and then slice the cone parallel to the $x$-axis, then the cross-sectional area of the slices

(a) Is constant  
(b) Increases as $y$ increases  
(c) Decreases as $y$ increases

6.2.4 True or False The volume of the solid of revolution is the same whether a region is revolved around the $x$-axis or the $y$-axis.

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident
6.2.5 Imagine taking the enclosed region in Figure 8.8 and rotating it about the $x$-axis. Which of the following graphs (a)-(d) represents the resulting volume as a function of $x$?

![Figure 8.8](image)

6.2.6 Imagine taking the enclosed region in Figure 8.9 and rotating it about the $y$-axis. Which of the following graphs (a)-(d) represents the resulting volume as a function of $x$?

![Figure 8.9](image)

6.2.7 Imagine that the region between the graphs of $f$ and $g$ in Figure 8.10 is rotated about the $x$-axis to form a solid. Which of the following represents the volume of this solid?
6.2. USING DEFINITE INTEGRALS TO FIND VOLUME

(a) $\int_0^q 2\pi x (f(x) - g(x)) dx$
(b) $\int_0^q (f(x) - g(x)) dx$
(c) $\int_0^q \pi (f(x) - g(x))^2 dx$
(d) $\int_0^q (\pi f^2(x) - \pi g^2(x)) dx$
(e) $\int_0^q \pi x (f(x) - g(x)) dx$

6.2.8 Imagine rotating the enclosed region in Figure 8.11 about three lines separately: the $x$-axis, the $y$-axis, and the vertical line at $x = 6$. This produces three different volumes. Which of the following lists those volumes in order from largest to smallest?

(a) $x$-axis; $x = 6$; $y$-axis
(b) $y$-axis; $x = 6$; $x$-axis
(c) $x = 6$; $y$-axis; $x$-axis
(d) $x = 6$; $x$-axis; $y$-axis
(e) $x$-axis; $y$-axis; $x = 6$
6.2. USING DEFINITE INTEGRALS TO FIND VOLUME

(f) \( y\)-axis; \( x\)-axis; \( x = 6 \)

6.2.9 Let \( R \) be the region bounded by the graph of \( f(x) = x \), the line \( x = 1 \), and the \( x\)-axis. True or false: The volume of the solid generated when \( R \) is revolved about the line \( x = 3 \) is given by \( \int_0^1 2\pi x(3 - x)\,dx \).

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

6.2.10 Let \( R \) be the region bounded by the graph of \( y = x \), the line \( y = 1 \), and the \( y\)-axis. If the shell method is used to determine the volume of the solid generated when \( R \) is revolved about the \( y\)-axis, what integral is obtained?

(a) \( \int_0^1 \pi y^2\,dy \)
(b) \( \int_0^1 2\pi x^2\,dx \)
(c) \( \int_0^1 2\pi x(1 - x)\,dx \)
(d) None of the above

6.2.11 The figure below shows the graph of \( y = (x + 1)^2 \) rotated around the \( x\)-axis.

The volume of the approximating slice that is \( x_i \) units away from the origin is

(a) \( \pi \frac{81x_i^2}{4} \Delta x \)
(b) $\pi x_i^2 \Delta x$
(c) $\pi (x_i + 1)^2 \Delta x$
(d) $\pi (x_i + 1)^4 \Delta x$

6.2.12 An integral that would calculate the volume of the solid obtained by revolving the graph of $y = (x + 1)^2$ from $x = 1$ to $x = 2$ around the $y$-axis is:

(a) $\int_1^2 \pi (\sqrt{x} - 1)^2 \, dx$
(b) $\int_1^9 \pi (x + 1)^4 \, dx$
(c) $\int_1^2 \pi (\sqrt{y} - 1)^2 \, dy$
(d) $\int_1^9 \pi (\sqrt{y} - 1)^2 \, dy$
6.3 Density, Mass, and Center of Mass

**Preview Activity 6.3.** In each of the following scenarios, we consider the distribution of a quantity along an axis.

(a) Suppose that the function $c(x) = 200 + 100e^{-0.1x}$ models the density of traffic on a straight road, measured in cars per mile, where $x$ is number of miles east of a major interchange, and consider the definite integral $\int_0^2 (200 + 100e^{-0.1x}) \, dx$.

i. What are the units on the product $c(x) \cdot \triangle x$?

ii. What are the units on the definite integral and its Riemann sum approximation given by

$$\int_0^2 c(x) \, dx \approx \sum_{i=1}^{n} c(x_i) \triangle x?$$

iii. Evaluate the definite integral $\int_0^2 c(x) \, dx = \int_0^2 (200 + 100e^{-0.1x}) \, dx$ and write one sentence to explain the meaning of the value you find.

(b) On a 6 foot long shelf filled with books, the function $B$ models the distribution of the weight of the books, measured in pounds per inch, where $x$ is the number of inches from the left end of the bookshelf. Let $B(x)$ be given by the rule $B(x) = 0.5 + \frac{1}{(x+1)^2}$.

i. What are the units on the product $B(x) \cdot \triangle x$?

ii. What are the units on the definite integral and its Riemann sum approximation given by

$$\int_{12}^{36} B(x) \, dx \approx \sum_{i=1}^{n} B(x_i) \triangle x?$$

iii. Evaluate the definite integral $\int_{12}^{36} B(x) \, dx = \int_{12}^{36} (0.5 + \frac{1}{(x+1)^2}) \, dx$ and write one sentence to explain the meaning of the value you find.

\(\checkmark\)
Activity 6.7.

Consider the following situations in which mass is distributed in a non-constant manner.

(a) Suppose that a thin rod with constant cross-sectional area of 1 cm$^2$ has its mass distributed according to the density function $\rho(x) = 2e^{-0.2x}$, where $x$ is the distance in cm from the left end of the rod, and the units on $\rho(x)$ are g/cm. If the rod is 10 cm long, determine the exact mass of the rod.

(b) Consider the cone that has a base of radius 4 m and a height of 5 m. Picture the cone lying horizontally with the center of its base at the origin and think of the cone as a solid of revolution.

i. Write and evaluate a definite integral whose value is the volume of the cone.
ii. Next, suppose that the cone has uniform density of 800 kg/m$^3$. What is the mass of the solid cone?
iii. Now suppose that the cone’s density is not uniform, but rather that the cone is most dense at its base. In particular, assume that the density of the cone is uniform across cross sections parallel to its base, but that in each such cross section that is a distance $x$ units from the origin, the density of the cross section is given by the function $\rho(x) = 400 + \frac{200}{1+x^2}$, measured in kg/m$^3$. Determine and evaluate a definite integral whose value is the mass of this cone of non-uniform density. Do so by first thinking about the mass of a given slice of the cone $x$ units away from the base; remember that in such a slice, the density will be essentially constant.

(c) Let a thin rod of constant cross-sectional area 1 cm$^2$ and length 12 cm have its mass be distributed according to the density function $\rho(x) = \frac{1}{25}(x - 15)^2$, measured in g/cm. Find the exact location $z$ at which to cut the bar so that the two pieces will each have identical mass.
Activity 6.8.

For quantities of equal weight, such as two children on a teeter-totter, the balancing point is found by taking the average of their locations. When the weights of the quantities differ, we use a weighted average of their respective locations to find the balancing point.

(a) Suppose that a shelf is 6 feet long, with its left end situated at \( x = 0 \). If one book of weight 1 lb is placed at \( x_1 = 0 \), and another book of weight 1 lb is placed at \( x_2 = 6 \), what is the location of \( \bar{x} \), the point at which the shelf would (theoretically) balance on a fulcrum?

(b) Now, say that we place four books on the shelf, each weighing 1 lb: at \( x_1 = 0 \), at \( x_2 = 2 \), at \( x_3 = 4 \), and at \( x_4 = 6 \). Find \( \bar{x} \), the balancing point of the shelf.

(c) How does \( \bar{x} \) change if we change the location of the third book? Say the locations of the 1-lb books are \( x_1 = 0 \), \( x_2 = 2 \), \( x_3 = 3 \), and \( x_4 = 6 \).

(d) Next, suppose that we place four books on the shelf, but of varying weights: at \( x_1 = 0 \) a 2-lb book, at \( x_2 = 2 \) a 3-lb book, and \( x_3 = 4 \) a 1-lb book, and at \( x_4 = 6 \) a 1-lb book. Use a weighted average of the locations to find \( \bar{x} \), the balancing point of the shelf. How does the balancing point in this scenario compare to that found in (b)?

(e) What happens if we change the location of one of the books? Say that we keep everything the same in (d), except that \( x_3 = 5 \). How does \( \bar{x} \) change?

(f) What happens if we change the weight of one of the books? Say that we keep everything the same in (d), except that the book at \( x_3 = 4 \) now weighs 2 lbs. How does \( \bar{x} \) change?

(g) Experiment with a couple of different scenarios of your choosing where you move the location of one of the books to the left, or you decrease the weight of one of the books.

(h) Write a couple of sentences to explain how adjusting the location of one of the books or the weight of one of the books affects the location of the balancing point of the shelf. Think carefully here about how your changes should be considered relative to the location of the balancing point \( \bar{x} \) of the current scenario.
Activity 6.9.

Consider a thin bar of length 20 cm whose density is distributed according to the function
\( \rho(x) = 4 + 0.1x \), where \( x = 0 \) represents the left end of the bar. Assume that \( \rho \) is measured in g/cm and \( x \) is measured in cm.

(a) Find the total mass, \( M \), of the bar.

(b) Without doing any calculations, do you expect the center of mass of the bar to be equal to 10, less than 10, or greater than 10? Why?

(c) Compute \( \bar{x} \), the exact center of mass of the bar.

(d) What is the average density of the bar?

(e) Now consider a different density function, given by \( p(x) = 4e^{0.020732x} \), also for a bar of length 20 cm whose left end is at \( x = 0 \). Plot both \( \rho(x) \) and \( p(x) \) on the same axes. Without doing any calculations, which bar do you expect to have the greater center of mass? Why?

(f) Compute the exact center of mass of the bar described in (e) whose density function is \( p(x) = 4e^{0.020732x} \). Check the result against the prediction you made in (e).
Voting Questions

6.3.1
6.4 Physics Applications: Work, Force, and Pressure

Preview Activity 6.4. A bucket is being lifted from the bottom of a 50-foot deep well; its weight (including the water), \( B \), in pounds at a height \( h \) feet above the water is given by the function \( B(h) \). When the bucket leaves the water, the bucket and water together weigh \( B(0) = 20 \) pounds, and when the bucket reaches the top of the well, \( B(50) = 12 \) pounds. Assume that the bucket loses water at a constant rate (as a function of height, \( h \)) throughout its journey from the bottom to the top of the well.

(a) Find a formula for \( B(h) \).

(b) Compute the value of the product \( B(5) \Delta h \), where \( \Delta h = 2 \) feet. Include units on your answer. Explain why this product represents the approximate work it took to move the bucket of water from \( h = 5 \) to \( h = 7 \).

(c) Is the value in (b) an over- or under-estimate of the actual amount of work it took to move the bucket from \( h = 5 \) to \( h = 7 \)? Why?

(d) Compute the value of the product \( B(22) \Delta h \), where \( \Delta h = 0.25 \) feet. Include units on your answer. What is the meaning of the value you found?

(e) More generally, what does the quantity \( W_{\text{slice}} = B(h) \Delta h \) measure for a given value of \( h \) and a small positive value of \( \Delta h \)?

(f) Evaluate the definite integral \( \int_0^{50} B(h) \, dh \). What is the meaning of the value you find? Why?
Activity 6.10.

Consider the following situations in which a varying force accomplishes work.

(a) Suppose that a heavy rope hangs over the side of a cliff. The rope is 200 feet long and weighs 0.3 pounds per foot; initially the rope is fully extended. How much work is required to haul in the entire length of the rope? (Hint: set up a function $F(h)$ whose value is the weight of the rope remaining over the cliff after $h$ feet have been hauled in.)

(b) A leaky bucket is being hauled up from a 100 foot deep well. When lifted from the water, the bucket and water together weigh 40 pounds. As the bucket is being hauled upward at a constant rate, the bucket leaks water at a constant rate so that it is losing weight at a rate of 0.1 pounds per foot. What function $B(h)$ tells the weight of the bucket after the bucket has been lifted $h$ feet? What is the total amount of work accomplished in lifting the bucket to the top of the well?

(c) Now suppose that the bucket in (b) does not leak at a constant rate, but rather that its weight at a height $h$ feet above the water is given by $B(h) = 25 + 15e^{-0.05h}$. What is the total work required to lift the bucket 100 feet? What is the average force exerted on the bucket on the interval $h = 0$ to $h = 100$?

(d) From physics, Hooke’s Law for springs states that the amount of force required to hold a spring that is compressed (or extended) to a particular length is proportionate to the distance the spring is compressed (or extended) from its natural length. That is, the force to compress (or extend) a spring $x$ units from its natural length is $F(x) = kx$ for some constant $k$ (which is called the spring constant.) For springs, we choose to measure the force in pounds and the distance the spring is compressed in feet. Suppose that a force of 5 pounds extends a particular spring 4 inches (1/3 foot) beyond its natural length.
   
   i. Use the given fact that $F(1/3) = 5$ to find the spring constant $k$.

   ii. Find the work done to extend the spring from its natural length to 1 foot beyond its natural length.

   iii. Find the work required to extend the spring from 1 foot beyond its natural length to 1.5 feet beyond its natural length.

<<
Activity 6.11.

In each of the following problems, determine the total work required to accomplish the described task. In parts (b) and (c), a key step is to find a formula for a function that describes the curve that forms the side boundary of the tank.

(a) Consider a vertical cylindrical tank of radius 2 meters and depth 6 meters. Suppose the tank is filled with 4 meters of water of mass density 1000 kg/m$^3$, and the top 1 meter of water is pumped over the top of the tank.

(b) Consider a hemispherical tank with a radius of 10 feet. Suppose that the tank is full to a depth of 7 feet with water of weight density 62.4 pounds/ft$^3$, and the top 5 feet of water are pumped out of the tank to a tanker truck whose height is 5 feet above the top of the tank.

(c) Consider a trough with triangular ends, as pictured in Figure 6.3, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft$^3$, and a pump is used to empty the tank until the water remaining in the tank is 1 foot deep.
Activity 6.12.
In each of the following problems, determine the total force exerted by water against the surface that is described.

(a) Consider a rectangular dam that is 100 feet wide and 50 feet tall, and suppose that water presses against the dam all the way to the top.

(b) Consider a semicircular dam with a radius of 30 feet. Suppose that the water rises to within 10 feet of the top of the dam.

(c) Consider a trough with triangular ends, as pictured in Figure 6.4, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft$^3$. How much force does the water exert against one of the triangular ends?
Voting Questions

6.4.1 **True or False** It takes more work to lift a 20-lb weight 10 feet slowly than to lift it the same distance quickly.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

6.4.2 A constant force of 5.2 lb pushes on an object, moving the object through a distance of 3 feet. Is an integral needed to determine how much work is done?

(a) Yes integral is needed
(b) No just need to multiply force by distance

6.4.3 A 3-lb book is lifted 5 feet off the floor. Is an integral needed to determine how much work is done?

(a) Yes - integral is needed
(b) No - just need to multiply force by distance

6.4.4 I’m carrying my garden hose around my yard, laying down hose to set up a path for a traveling sprinkler. Is an integral needed to determine how much work is done?

(a) Yes integral is needed
(b) No just need to multiply force by distance

6.4.5 The average value of the force, $F(x)$, exerted on an object while moving the object over the interval $1 \leq x \leq 4$ is 7 N. Is an integral needed to determine how much work is done?

(a) Yes integral is needed
(b) No just need to multiply force by distance

6.4.6 I’m pushing a shopping cart around the grocery store, filling it with my groceries. Is an integral needed to determine how much work is done on the shopping cart?
6.4.7 My boat is floating 20 feet offshore, and I use a rope to pull it in to the beach. I pull on the rope with a constant force, but the boat moves faster and faster as it gets closer to the beach so its distance from the shore is given by the function \( d(t) = 20 - 3t^2 \), where \( t \) is in seconds. Is an integral needed to determine how much work is done on the boat? (ignore the weight of the rope)

(a) Yes - integral is needed
(b) No just need to multiply force by distance

6.4.8 You are lifting a 15 kg bucket 3 meters up from the ground to the second floor of a building. The bucket is held by a heavy chain that has a mass of 2 kg per meter, so the farther up you lift it, the easier it becomes, because there is less chain out. Recall that the force of gravity (in Newtons) is equal to mass (in kg) times \( g \) (in m/s\(^2\)), and assume that \( g \approx 10 \text{ m/s}^2 \). How much work does it take to raise the bucket?

(a) 45 Joules
(b) 90 Joules
(c) 450 Joules
(d) 540 Joules
(e) 630 Joules
(f) None of the above
6.5 Improper Integrals

Preview Activity 6.5. A company with a large customer base has a call center that receives thousands of calls a day. After studying the data that represents how long callers wait for assistance, they find that the function \( p(t) = 0.25e^{-0.25t} \) models the time customers wait in the following way: the fraction of customers who wait between \( t = a \) and \( t = b \) minutes is given by

\[
\int_{a}^{b} p(t) \, dt.
\]

Use this information to answer the following questions.

(a) Determine the fraction of callers who wait between 5 and 10 minutes.

(b) Determine the fraction of callers who wait between 10 and 20 minutes.

(c) Next, let’s study how the fraction who wait up to a certain number of minutes:
   i. What is the fraction of callers who wait between 0 and 5 minutes?
   ii. What is the fraction of callers who wait between 0 and 10 minutes?
   iii. Between 0 and 15 minutes? Between 0 and 20?

(d) Let \( F(b) \) represent the fraction of callers who wait between 0 and \( b \) minutes. Find a formula for \( F(b) \) that involves a definite integral, and then use the First FTC to find a formula for \( F(b) \) that does not involve a definite integral.

(e) What is the value of \( \lim_{b \to \infty} F(b) \)? Why?
Activity 6.13.

In this activity we explore the improper integrals \( \int_{1}^{\infty} \frac{1}{x} \, dx \) and \( \int_{1}^{\infty} \frac{1}{x^{3/2}} \, dx \).

(a) First we investigate \( \int_{1}^{\infty} \frac{1}{x} \, dx \).

i. Use the First FTC to determine the exact values of \( \int_{1}^{10} \frac{1}{x} \, dx \), \( \int_{1}^{1000} \frac{1}{x} \, dx \), and \( \int_{1}^{100000} \frac{1}{x} \, dx \). Then, use your calculator to compute a decimal approximation of each result.

ii. Use the First FTC to evaluate the definite integral \( \int_{1}^{b} \frac{1}{x} \, dx \) (which results in an expression that depends on \( b \)).

iii. Now, use your work from (ii.) to evaluate the limit given by

\[
\lim_{b \to \infty} \int_{1}^{b} \frac{1}{x} \, dx.
\]

(b) Next, we investigate \( \int_{1}^{\infty} \frac{1}{x^{3/2}} \, dx \).

i. Use the First FTC to determine the exact values of \( \int_{1}^{10} \frac{1}{x^{3/2}} \, dx \), \( \int_{1}^{1000} \frac{1}{x^{3/2}} \, dx \), and \( \int_{1}^{100000} \frac{1}{x^{3/2}} \, dx \). Then, use your calculator to compute a decimal approximation of each result.

ii. Use the First FTC to evaluate the definite integral \( \int_{1}^{b} \frac{1}{x^{3/2}} \, dx \) (which results in an expression that depends on \( b \)).

iii. Now, use your work from (ii.) to evaluate the limit given by

\[
\lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^{3/2}} \, dx.
\]

(c) Plot the functions \( y = \frac{1}{x} \) and \( y = \frac{1}{x^{3/2}} \) on the same coordinate axes for the values \( x = 0 \ldots 10 \). How would you compare their behavior as \( x \) increases without bound? What is similar? What is different?

(d) How would you characterize the value of \( \int_{1}^{\infty} \frac{1}{x} \, dx \)? of \( \int_{1}^{\infty} \frac{1}{x^{3/2}} \, dx \)? What does this tell us about the respective areas bounded by these two curves for \( x \geq 1 \)?
Determine whether each of the following improper integrals converges or diverges. For each integral that converges, find its exact value.

(a) $\int_1^\infty \frac{1}{x^2} \, dx$

(b) $\int_0^\infty e^{-x/4} \, dx$

(c) $\int_2^\infty \frac{9}{(x + 5)^{2/3}} \, dx$

(d) $\int_4^\infty \frac{3}{(x + 2)^{5/4}} \, dx$

(e) $\int_0^\infty xe^{-x/4} \, dx$

(f) $\int_1^\infty \frac{1}{x^p} \, dx$, where $p$ is a positive real number
Activity 6.15.

For each of the following definite integrals, decide whether the integral is improper or not. If the integral is proper, evaluate it using the First FTC. If the integral is improper, determine whether or not the integral converges or diverges; if the integral converges, find its exact value.

(a) \( \int_0^1 \frac{1}{x^{1/3}} \, dx \)
(b) \( \int_0^2 e^{-x} \, dx \)
(c) \( \int_1^4 \frac{1}{\sqrt{1-x}} \, dx \)
(d) \( \int_{-2}^2 \frac{1}{x^2} \, dx \)
(e) \( \int_0^{\pi/2} \tan(x) \, dx \)
(f) \( \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx \)
Voting Questions

6.5.1 **True or False:** If $f$ is continuous for all $x$ and \( \int_0^\infty f(x)\,dx \) converges, then so does \( \int_a^\infty f(x)\,dx \) for all positive $a$.

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident

6.5.2 **True or False:** If $f$ is continuous for all $x$ and \( \int_0^\infty f(x)\,dx \) diverges, then so does \( \int_a^\infty f(x)\,dx \) for all positive $a$.

(a) True, and I am very confident  
(b) True, but I am not very confident  
(c) False, but I am not very confident  
(d) False, and I am very confident

6.5.3 **Does** \( \int_1^\infty \frac{dx}{1+x^2} \)

(a) Converge  
(b) Diverge  
(c) Can’t tell with what we know

6.5.4 **Does** \( \int_1^\infty \frac{dx}{\sqrt{x^4+x^2+1}} \)

(a) Converge  
(b) Diverge  
(c) Can’t tell with what we know

6.5.5 **Does** \( \int_2^\infty \frac{dx}{x^2-1} \)

(a) Converge by direct comparison with \( \int_2^\infty \frac{1}{x^2}\,dx \)  
(b) Diverge by direct comparison with \( \int_2^\infty \frac{1}{x^2}\,dx \)  
(c) Can’t tell by direct comparison with \( \int_2^\infty \frac{1}{x^2}\,dx \)
6.5. IMPROPER INTEGRALS

6.5.6 Is this an improper integral?
\[\int_{1}^{\infty} \frac{\sin x}{x} \, dx\]
(a) Yes, it is improper.
(b) No, it is proper.

6.5.7 Is this an improper integral?
\[\int_{1}^{5} \frac{1}{x} \, dx\]
(a) Yes, it is improper.
(b) No, it is proper.

6.5.8 Is this an improper integral?
\[\int_{0}^{1} \frac{1}{2 - 3x} \, dx\]
(a) Yes, it is improper.
(b) No, it is proper.

6.5.9 Is this an improper integral?
\[\int_{3}^{4} \frac{1}{\sin x} \, dx\]
(a) Yes, it is improper.
(b) No, it is proper.

6.5.10 Is this an improper integral?
\[\int_{-3}^{3} x^{-1/3} \, dx\]
(a) Yes, it is improper.
(b) No, it is proper.

6.5.11 Is this an improper integral?
\[\int_{1}^{2} \frac{1}{2x - 1} \, dx\]
(a) Yes, it is improper.
(b) No, it is proper.

6.5.12 Is this an improper integral?

\[ \int_{1}^{2} \ln(x - 1) \, dx \]

(a) Yes, it is improper.
(b) No, it is proper.
Chapter 7

Differential Equations

7.1 An Introduction to Differential Equations

Preview Activity 7.1. The position of a moving object is given by the function $s(t)$, where $s$ is measured in feet and $t$ in seconds. We determine that the velocity is $v(t) = 4t + 1$ feet per second.

(a) How much does the position change over the time interval $[0, 4]$?

(b) Does this give you enough information to determine $s(4)$, the position at time $t = 4$? If so, what is $s(4)$? If not, what additional information would you need to know to determine $s(4)$?

(c) Suppose you are told that the object’s initial position $s(0) = 7$. Determine $s(2)$, the object’s position 2 seconds later.

(d) If you are told instead that the object’s initial position is $s(0) = 3$, what is $s(2)$?

(e) If we only know the velocity $v(t) = 4t + 1$, is it possible that the object’s position at all times is $s(t) = 2t^2 + t - 4$? Explain how you know.

(f) Are there other possibilities for $s(t)$? If so, what are they?

(g) If, in addition to knowing the velocity function is $v(t) = 4t + 1$, we know the initial position $s(0)$, how many possibilities are there for $s(t)$?
Activity 7.1.

Express the following statements as differential equations. In each case, you will need to introduce notation to describe the important quantities in the statement.

(a) The population of a town grows at an annual rate of 1.25%.
(b) A radioactive sample losses 5.6% of its mass every day.
(c) You have a bank account that earns 4% interest every year. At the same time, you withdraw money continually from the account at the rate of $1000 per year.
(d) A cup of hot chocolate is sitting in a 70° room. The temperature of the hot chocolate cools by 10% of the difference between the hot chocolate’s temperature and the room temperature every minute.
(e) A can of cold soda is sitting in a 70° room. The temperature of the soda warms at the rate of 10% of the difference between the soda’s temperature and the room’s temperature every minute.
Activity 7.2.

Shown below are two graphs depicting the velocity of falling objects. On the left is the velocity of a skydiver, while on the right is the velocity of a meteorite entering the Earth’s atmosphere.

(a) Begin with the skydiver’s velocity and measure the rate of change $dv/dt$ when the velocity is $v = 0.5, 1.0, 1.5, 2.0,$ and $2.5$. Plot your values on the graph below. You will want to think carefully about this: you are plotting the derivative $dv/dt$ as a function of velocity.
(b) Now do the same thing with the meteorite’s velocity: measure the rate of change $\frac{dv}{dt}$ when the velocity is $v = 3.5, 4.0, 4.5$, and $5.0$. Plot your values on the graph above.

(c) You should find that all your point lie on a line. Write the equation of this line being careful to use proper notation for the quantities on the horizontal and vertical axes.

(d) The relationship you just found is a differential equation. Write a complete sentence that explains its meaning.

(e) By looking at the differential equation, determine the values of the velocity for which the velocity increases?

(f) By looking at the differential equation, determine the values of the velocity for which the velocity decreases?

(g) By looking at the differential equation, determine the values of the velocity for which the velocity remains constant.
7.1. AN INTRODUCTION TO DIFFERENTIAL EQUATIONS

Voting Questions

7.1.1 Which of the following is not a differential equation?

(a) \( y' = 3y \)

(b) \( 2x^2y + y^2 = 6 \)

(c) \( tx \frac{dx}{dt} = 2 \)

(d) \( \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 7y + 8x = 0 \)

(e) All are differential equations.

7.1.2 Which of the following is not a differential equation?

(a) \( 6 \frac{dy}{dx} + 3xy \)

(b) \( 8 = \frac{y'}{y} \)

(c) \( 2 \frac{d^2f}{dx^2} + 7 \frac{df}{dx} = f \)

(d) \( h(x) + 2h'(x) = g(x) \)

(e) All are differential equations.

7.1.3 Which of the following couldn’t be the solution of a differential equation?

(a) \( z(t) = 6 \)

(b) \( y = 3x^2 + 7 \)

(c) \( x = 0 \)

(d) \( y = 3x + y' \)

(e) All could be solutions of a differential equation.

7.1.4 Which of the following could not be a solution of a differential equation?
7.1.5 Which of the following could not be a solution of a differential equation?

(a) \( f = 2y + 7 \)
(b) \( q(d) = 2d^2 - 6e^d \)
(c) \( 6y^2 + 2yx = \sqrt{x} \)
(d) \( y = 4 \sin 8\pi z \)
(e) All could be a solution of a differential equation.

7.1.6 True or False? A differential equation is a type of function.

(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

7.1.7 Suppose \( \frac{dx}{dt} = 0.5x \) and \( x(0) = 8 \). Then the value of \( x(2) \) is approximately

(a) 4
7.1. AN INTRODUCTION TO DIFFERENTIAL EQUATIONS

(b) 8
(c) 9
(d) 12
(e) 16

7.1.8 Which of the following is a solution to the differential equation \( \frac{dy}{dt} = 72 - y \)?

(a) \( y(t) = 72t - \frac{1}{2}t^2 \)
(b) \( y(t) = 72 + e^{-t} \)
(c) \( y(t) = e^{-72t} \)
(d) \( y(t) = e^{-t} \)

7.1.9 The amount of a chemical in a lake is decreasing at a rate of 30% per year. If \( p(t) \) is the total amount of the chemical in the lake as a function of time \( t \) (in years), which differential equation models this situation?

(a) \( p'(t) = -30 \)
(b) \( p'(t) = -0.30 \)
(c) \( p'(t) = p - 30 \)
(d) \( p'(t) = -0.3p \)
(e) \( p'(t) = 0.7p \)

7.1.10 The evolution of the temperature of a hot cup of coffee cooling off in a room is described by \( \frac{dT}{dt} = -0.01T + 0.6 \), where \( T \) is in °F and \( t \) is in hours. What are the units of the numbers -0.01 and 0.6?

(a) -0.01 °F, and 0.6 °F
(b) -0.01 per hour, and 0.6 °F per hour
(c) -0.01 °F per hour, and 0.6 °F
(d) neither number has units

7.1.11 We want to test the function \( z(x) = 4 \sin 3x \) to see if it solves \( z'' + 2z' + 4z = 0 \), by substituting the function into the differential equation. What is the resulting equation before simplification?

(a) \(-36 \sin 3x + 24 \cos 3x + 16 \sin 3x = 0\)
7.1. AN INTRODUCTION TO DIFFERENTIAL EQUATIONS

(b) \(4 \sin 3x + 8 \sin 3x + 16 \sin 3x = 0\)
(c) \(-36 \sin 3x + 12 \cos 3x + 4 \sin 3x = 0\).
(d) \(4 \sin 3x + 8 \cos 3x + 4 \sin 3x = 0\)
(e) none of the above

7.1.12 If we test the function \(f(x) = ae^{bx}\) to see if it could solve \(\frac{df}{dx} = cf^2\), which equation is the result?

(a) \(\frac{df}{dx} = ca^2 e^{2bx}\)
(b) \(abe^{bx} = cf^2\)
(c) \(ae^{bx} = ca^2 e^{(bx)^2}\)
(d) \(abe^{bx} = ca e^{2bx}\)
(e) \(abe^{bx} = cae^{bx}\)
(f) None of the above

7.1.13 We want to test the function \(f(x) = 3e^{2x} + 6x\) to see if it solves the differential equation \(\frac{df}{dx} = 2f + 3x\), so we insert the function and its derivative, getting \(6e^{2x} + 6 = 2(3e^{2x} + 6x) + 3x\). This means that:

(a) This function is a solution.
(b) This function is a solution if \(x = 2/5\).
(c) This function is not a solution.
(d) Not enough information is given.

7.1.14 A bookstore is constantly discarding a certain percentage of its unsold inventory and also receiving new books from its supplier so that the rate of change of the number of books in inventory is \(B'(t) = -0.02B + 400 + 0.05t\), where \(B\) is the number of books and \(t\) is in months. If the store begins with 10,000 books in inventory, at what rate is it receiving books from its supplier at \(t = 0\)?

(a) 200 books per month
(b) 400 books per month
(c) -200 books per month
(d) 900 books per month
7.2 Qualitative behavior of solutions to differential equations

Preview Activity 7.2. Let’s consider the initial value problem

\[
\frac{dy}{dt} = t - 2, \quad y(0) = 1.
\]

(a) Use the differential equation to find the slope of the tangent line to the solution \( y(t) \) at \( t = 0 \). Then use the initial value to find the equation of the tangent line at \( t = 0 \). Sketch this tangent line over the interval \(-0.25 \leq t \leq 0.25\) on the axes provided.

(b) Also shown in the given figure are the tangent lines to the solution \( y(t) \) at the points \( t = 1, 2, \) and \( 3 \) (we will see how to find these later). Use the graph to measure the slope of each tangent line and verify that each agrees with the value specified by the differential equation.

(c) Using these tangent lines as a guide, sketch a graph of the solution \( y(t) \) over the interval \( 0 \leq t \leq 3 \) so that the lines are tangent to the graph of \( y(t) \).

(d) Use the Fundamental Theorem of Calculus to find \( y(t) \), the solution to this initial value problem.

(e) Graph the solution you found in (d) on the axes provided, and compare it to the sketch you made using the tangent lines.
Activity 7.3.

Let’s consider the differential equation

\[
\frac{dy}{dt} = -\frac{1}{2}(y - 4).
\]

(a) Make a plot of \( \frac{dy}{dt} \) versus \( y \). Looking at the graph, for what values of \( y \) does \( y \) increase and for what values does it decrease?

(b) Now sketch the slope field for this differential equation.

(c) Sketch the solutions that satisfy \( y(0) = 0 \), \( y(0) = 2 \), \( y(0) = 4 \) and \( y(0) = 6 \).

(d) Verify that \( y(t) = 4 + 2e^{-t/2} \) is a solution to the differential equation with the initial value \( y(0) = 6 \). Compare its graph to the one you sketched above.

(e) What is special about the solution where \( y(0) = 4 \)?
Activity 7.4.

Let’s consider the differential equation

\[ \frac{dy}{dt} = -\frac{1}{2}y(y - 4). \]

(a) Make a plot of \( \frac{dy}{dt} \) versus \( y \). Looking at the graph, for what values of \( y \) does \( y \) increase and for what values does it decrease?

(b) Make a phase line plot of \( y \) showing the regions where \( y \) is increasing and decreasing.

(c) Identify any equilibrium solutions of this differential equation using both the plot of \( \frac{dy}{dt} \) vs \( y \) and the phase line plot.

(d) Now sketch the slope field for this differential equation.

(e) Sketch the solutions with an initial value in the range \(-1 \leq y(0) \leq 5\).

(f) An equilibrium solution \( \overline{y} \) is called *stable* if nearby solutions converge to \( \overline{y} \). This means that if the initial condition varies slightly from \( \overline{y} \), then \( \lim_{t \to \infty} y(t) = \overline{y} \).

Conversely, an equilibrium solution \( \bar{y} \) is called *unstable* if nearby solutions are pushed away from \( \bar{y} \).
Using your work above, classify the equilibrium solutions you found as either stable or unstable.

(g) Suppose that \( y(t) \) describes the population of a species of living organisms and that the initial value \( y(0) > 0 \). What can you say about the eventual fate of this population?

(h) Remember that an equilibrium solution \( \bar{y} \) satisfies \( f(\bar{y}) = 0 \). If we graph \( \frac{dy}{dt} = f(y) \) as a function of \( y \), for which of the following differential equations is \( \bar{y} \) a stable equilibrium and for which is it unstable?
7.2. QUALITATIVE BEHAVIOR OF SOLUTIONS TO DIFFERENTIAL EQUATIONS

Voting Questions

7.2.1 What does the differential equation \( \frac{dy}{dx} = 2y \) tell us about the slope of the solution curves at any point?

(a) The slope is always 2.
(b) The slope is equal to the \( x \)-coordinate.
(c) The slope is equal to the \( y \)-coordinate.
(d) The slope is equal to two times the \( x \)-coordinate.
(e) The slope is equal to two times the \( y \)-coordinate.
(f) None of the above.

7.2.2 The slopefield below indicates that the differential equation has which form?

(a) \( \frac{dy}{dt} = f(y) \)
(b) \( \frac{dy}{dt} = f(t) \)
(c) \( \frac{dy}{dt} = f(y, t) \)

7.2.3 The slopefield below indicates that the differential equation has which form?
7.2. Qualitative Behavior of Solutions to Differential Equations

7.2.4 The slopefield below indicates that the differential equation has which form?

- (a) $\frac{dy}{dt} = f(y)$
- (b) $\frac{dy}{dt} = f(t)$
- (c) $\frac{dy}{dt} = f(y, t)$
7.2.5 The arrows in the slope field below have slopes that match the derivative \( y' \) for a range of values of the function \( y \) and the independent variable \( t \). Suppose that \( y(0) = 0 \). What would you predict for \( y(5) \)?

(a) \( y(5) \approx -3 \)
(b) \( y(5) \approx +3 \)
(c) \( y(5) \approx 0 \)
(d) \( y(5) < -5 \)
(e) None of the above

7.2.6 The arrows in the slope field below give the derivative \( y' \) for a range of values of the function \( y \) and the independent variable \( t \). Suppose that \( y(0) = -4 \). What would you predict for \( y(5) \)?
7.2.7 The slope field below represents which of the following differential equations?

(a) \( y(5) \approx -3 \)

(b) \( y(5) \approx +3 \)

(c) \( y(5) \approx 0 \)

(d) \( y(5) < -5 \)

(e) None of the above
7.2. Qualitative Behavior of Solutions to Differential Equations

7.2.8 Consider the differential equation $y' = ay + b$ with parameters $a$ and $b$. To approximate this function using Euler's method, what difference equation would we use?

(a) $y_{n+1} = ay_n + b$
(b) $y_{n+1} = \frac{y_n}{t}$
(c) $y_{n+1} = -yt$
(d) $y_{n+1} = -y_t$
(e) None of the above

7.2.9 Which of the following is the slope field for $\frac{dy}{dx} = x + y$?

(a) $y' = yt$
(b) $y' = \frac{y}{t}$
(c) $y' = -yt$
(d) $y' = -\frac{y}{t}$

7.2.8 Consider the differential equation $y' = ay + b$ with parameters $a$ and $b$. To approximate this function using Euler’s method, what difference equation would we use?

(a) $y_{n+1} = ay_n + b$
(b) $y_{n+1} = y_n + ay_n \Delta t + b\Delta t$
(c) $y_{n+1} = ay_n \Delta t + b\Delta t$
(d) $y_{n+1} = y_n \Delta t + ay_n \Delta t + b\Delta t$
(e) None of the above

7.2.9 Which of the following is the slope field for $\frac{dy}{dx} = x + y$?
Below is the slope field for \( \frac{dy}{dx} = y(1 - y) \):
As \( x \to \infty \), the solution to this differential equation that satisfies the initial condition \( y(0) = 2 \) will

(a) Increase asymptotically to \( y = 1 \)
(b) Decrease asymptotically to \( y = 1 \)
(c) Increase without bound
(d) Decrease without bound
(e) Start and remain horizontal
7.3 Euler’s method

Preview Activity 7.3. Consider the initial value problem

\[ \frac{dy}{dt} = \frac{1}{2}(y + 1), \quad y(0) = 0. \]

(a) Use the differential equation to find the slope of the tangent line to the solution \( y(t) \) at \( t = 0 \). Then use the given initial value to find the equation of the tangent line at \( t = 0 \).

(b) Sketch the tangent line on the axes below on the interval \( 0 \leq t \leq 2 \) and use it to approximate \( y(2) \), the value of the solution at \( t = 2 \).

(c) Assuming that your approximation for \( y(2) \) is the actual value of \( y(2) \), use the differential equation to find the slope of the tangent line to \( y(t) \) at \( t = 2 \). Then, write the equation of the tangent line at \( t = 2 \).

(d) Add a sketch of this tangent line to your plot on the axes above on the interval \( 2 \leq t \leq 4 \); use this new tangent line to approximate \( y(4) \), the value of the solution at \( t = 4 \).

(e) Repeat the same step to find an approximation for \( y(6) \).
Activity 7.5.

(a) Consider the initial value problem:

\[ \frac{dy}{dt} = 2t - 1 \]
\[ y(0) = 0 \]

Use Euler’s method with \( \Delta t = 0.2 \) to approximate the solution at \( t_i = 0.2, 0.4, 0.6, 0.8, \) and 1.0. Sketch the points \((t_i, y_i)\) on the right.

(b) Find the exact solution to the initial value problem in part a) and find the error in your approximation at each one of the points \( t_i \).

(c) Explain why Euler’s method for this initial value problem produces a left Riemann sum for the definite integral \( \int_0^1 (2t - 1) \, dt \).

(d) How would your result differ if the initial value was \( y(0) = 1 \)?
Activity 7.6.

Let’s consider the differential equation:

\[ \frac{dy}{dt} = 6y - y^2 \]

(a) Sketch the slope field for this differential equation below.

(b) Identify any equilibrium solutions and determine whether they are stable or unstable.

(c) What is the long-term behavior of the solution with initial value \( y(0) = 1 \).

(d) Using the initial value \( y(0) = 1 \), use Euler’s method with \( \Delta t = 0.2 \) to approximate the solution at \( t_i = 0.2, 0.4, 0.6, 0.8, \) and 1.0. Sketch the points \((t_i, y_i)\) on the right.

<table>
<thead>
<tr>
<th>( t_i )</th>
<th>( y_i )</th>
<th>( dy/dt )</th>
<th>( \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(e) What happens if you apply Euler’s method to approximate the solution with \( y(0) = 6 \)?
Exercises

1. We have seen that the error in approximating the solution to an initial value problem is proportional to $\Delta t$. That is, if $E_{\Delta t}$ is the Euler’s method approximation to the solution to an initial value problem at $t^*$, then

$$y(t^*) - E_{\Delta t} \approx K\Delta t$$

for some constant of proportionality $K$.

In this problem, we will see how to use this fact to improve our estimates, using an idea called accelerated convergence.

(a) We will create a new approximation by assuming the error is **exactly** proportional to $\Delta t$, according to the formula

$$y(t^*) - E_{\Delta t} = K\Delta t.$$ 

Using our earlier results from the initial value problem $\frac{dy}{dt} = y$ and $y(0) = 1$ with $\Delta t = 0.2$ and $\Delta t = 0.1$, we have

$$y(1) - 2.4883 = 0.2K$$

$$y(1) - 2.5937 = 0.1K.$$ 

This is a system of two linear equations in the unknowns $y(1)$ and $K$. Solve this system to find a new approximation for $y(1)$. (You may remember that the exact value is $y(1) = e = 2.71828\ldots$)

(b) Use the other data, $E_{0.05} = 2.6533$ and $E_{0.025} = 2.6851$ to do similar work as in (a) to obtain another approximation. Which gives the better approximation? Why do you think this is?

(c) Let’s now study the initial value problem

$$\frac{dy}{dt} = t - y, \quad y(0) = 0.$$ 

Approximate $y(0.3)$ by applying Euler’s method to find approximations $E_{0.1}$ and $E_{0.05}$. Now use the idea of accelerated convergence to obtain a better approximation. (For the sake of comparison, you want to note that the actual value is $y(0.3) = 0.0408$.)

2. In this problem, we’ll modify Euler’s method to obtain better approximations to solutions of initial value problems. This method is called the **Improved Euler’s method**.

In Euler’s method, we walk across an interval of width $\Delta t$ using the slope obtained from the differential equation at the left endpoint of the interval. Of course, the slope of the solution will most likely change over this interval. We can improve our approximation by trying to incorporate the change in the slope over the interval.

Let’s again consider the initial value problem $\frac{dy}{dt} = y$ and $y(0) = 1$, which we will approximate using steps of width $\Delta t = 0.2$. Our first interval is therefore $0 \leq t \leq 0.2$. At $t = 0$, the
differential equation tells us that the slope is 1, and the approximation we obtain from Euler’s method is that \( y(0.2) \approx y_1 = 1 + 1(0.2) = 1.2 \).

This gives us some idea for how the slope has changed over the interval \( 0 \leq t \leq 0.2 \). We know the slope at \( t = 0 \) is 1, while the slope at \( t = 0.2 \) is 1.2, trusting in the Euler’s method approximation. We will therefore refine our estimate of the initial slope to be the average of these two slopes; that is, we will estimate the slope to be \( (1 + 1.2)/2 = 1.1 \). This gives the new approximation \( y(1) = y_1 = 1 + 1.1(0.2) = 1.22 \).

The first few steps look like this:

<table>
<thead>
<tr>
<th>( t_i )</th>
<th>( y_i )</th>
<th>Slope at ((t_{i+1}, y_{i+1}))</th>
<th>Average slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000</td>
<td>1.2000</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2200</td>
<td>1.4640</td>
<td>1.3420</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4884</td>
<td>1.7861</td>
<td>1.6372</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

(a) Continue with this method to obtain an approximation for \( y(1) = e \).
(b) Repeat this method with \( \Delta t = 0.1 \) to obtain a better approximation for \( y(1) \).
(c) We saw that the error in Euler’s method is proportional to \( \Delta t \). Using your results from parts (a) and (b), what power of \( \Delta t \) appears to be proportional to the error in the Improved Euler’s Method?

3. In this problem we’ll walk you through how to set up Euler’s method in a spreadsheet program like Excel. We will then use it to solve several initial value problems. We’ll walk you through all of the steps for a simple problem first, then we’ll solve and interpret some real problems.

Implementing Euler’s Method in Excel:
In this example we’re going to examine the initial value problem

\[
\frac{dy}{dt} = ky + t, \quad \text{where} \quad y(0) = 1 \quad \text{and} \quad k = -1 \quad \text{is a problem parameter.} \quad (7.1)
\]

Remember that Euler’s Method finds the new \( y \) value of the solution by

\[
y_{\text{new}} = y_{\text{old}} + \frac{dy}{dt} \cdot \Delta t. \quad (7.2)
\]

The value of \( \frac{dy}{dt} \) is given from the differential equation.

- Open Excel
- Set up your Excel file as shown in Figure 7.1. Column A will be the time and column B will be the \( y \)-values. Cells E1 and E2 are parameters for the problem. Cell A2 is the initial time, and cell B2 is the initial value given in equation (7.1).
- Column A is the list of all of the times. To get this column started we define A3 by
  \[=A2 + \$E\$1\]
  Now fill this cell down to get a list of times ending at 3.
In cell B3 you need to code Euler’s method for the differential equation (7.1):

$$y_{\text{old}} + \frac{dy}{dt} \Delta t$$

where $dy/dt = k$.

Select cells A3 and B3 and fill them down to the end of the times.

For the problem defined in equation (7.1) a solution plot for $0 \leq t \leq 3$ is shown in Figure 7.2 for $k = -1$ and $k = -2$. These are the typical types of solution plots that arises from Euler’s method.

Solutions to Equation (7.1)

![Solutions to Equation (7.1)](image)

Figure 7.2: Euler’s method with $\Delta t = 0.1$ for equation (7.1) with different values of the parameter $k$. 
Newton's Law of Cooling says that the rate at which an object, such as a cup of coffee, cools is proportional to the difference in the object's temperature and room temperature. If $T(t)$ is the object's temperature and $T_r$ is room temperature, this law is expressed at

$$\frac{dT}{dt} = -k(T - T_r),$$

where $k$ is a constant of proportionality. In this problem, temperature is measured in degrees Fahrenheit and time in minutes.

(i) Two calculus students, Alice and Bob, enter a 70° classroom at the same time. Each has a cup of coffee that is 100°. The differential equation for Alice has a constant of proportionality $k = 0.5$, while the constant of proportionality for Bob is $k = 0.1$.

What is the initial rate of change for Alice's coffee? What is the initial rate of change for Bob's coffee?

(ii) What feature of Alice's and Bob's cups of coffee could explain this difference?

(iii) As the heating unit turns on and off in the room, the temperature in the room is $T_r = 70 + 10 \sin t$.

Implement Euler's method in Excel with a step size of $\Delta t = 0.1$ to approximate the temperature of Alice's coffee over the time interval $0 \leq t \leq 50$. Graph the temperature of her coffee and room temperature over this interval.

(iv) In the same way, implement Euler's method to approximate the temperature of Bob's coffee over the same time interval. Graph the temperature of his coffee and room temperature over the interval.

(v) Explain the similarities and differences that you see in the behavior of Alice's and Bob's cups of coffee.

(b) Contamination of a Lake

Suppose a lake has a river running through it and a factory built next to the lake begins releasing a chemical into the lake. The water running in to the lake is free of the chemical, and initially the lake is free of the chemical. The rate at which the amount of chemical in the lake increases ($dy/dt$) is equal to the rate at which the chemical enters the lake minus the rate at which the chemical leaves the lake. Furthermore, the rate at which the chemical leaves the lake is the product of the concentration of the chemical in the lake and the rate at which water leaves the lake.

To get an equation we need some notation. Let $V$ (m$^3$) be the volume of the lake. The phrase “A river running through the lake” means that there is a river running in to the
lake at a rate $R \text{ m}^3/\text{day}$ and a river running out of the lake also at a rate $R \text{ m}^3/\text{day}$. Assume the factory releases the chemical into the lake at a rate of $T \text{ kg/day}$. Finally, let $y(t)$ be the kilograms of chemical in the lake $t$ days after the factory begins dumping the chemicals.

(i) What is the differential equation that models this situation? Fill in the blanks in the statement of the model. Be sure to check the units carefully.

\[
\frac{dy}{dt} = \square - \square \cdot \square
\]

(ii) What is a proper initial condition? $y(0) =$?

(iii) Use Euler’s Method to solve the initial value problem from parts (i) and (ii) under the following circumstances. In each case assume that the factory dumps chemicals into the lake at a rate of $T = 100 \text{ kg/day}$ and that the volume of the lake is $V = 10,000 \text{ m}^3$.

(I) If the river is flowing at a rate of $R = 1,000 \text{ m}^3/\text{day}$, how does the mass of chemical in the lake evolve over a 60 day time period? Show a plot and discuss and observations. What is the concentration of the chemical in the lake after the 60 days?

\[
\text{Concentration} = \frac{\text{Mass}}{\text{Volume}}
\]

(II) Now assume that the river is flowing at a rate of $R = 2,000 \text{ m}^3/\text{day}$, how does the mass of chemical in the lake evolve over a 60 day time period? Show a plot and discuss and observations. What is the concentration of the chemical in the lake after the 60 days?

(c) Chemical Kinetics

Chemical kinetics is the study of the rates of chemical reactions. A reaction that occurs by the collision and combination of two molecules $A$ and $B$,

\[
A + B \rightarrow AB
\]

has a reaction rate that is proportional to the concentrations of $A$ and $B$. For the sake of notation, let $[A]$ be the concentration of $A$ and let $[B]$ be the concentration of $B$. The differential equation is

\[
\frac{d[A]}{dt} = k[A][B]. \quad (7.3)
\]

In the event that the concentrations of $A$ and $B$ are equal, equation (7.3) becomes

\[
\frac{d[A]}{dt} = k[A]^2. \quad (7.4)
\]

Consider equal quantities of gaseous hydrogen and iodine mixed in the reaction

\[
H_2 + I_2 \rightarrow 2HI
\]

to form gaseous Hydrogen iodide. Let $y(t)$ be the concentrations $[H_2] = [I_2]$. 

\[
\frac{d[A]}{dt} = k[A]^2.
\]
(i) Write the differential equation for the rate of reaction of equal quantities of gaseous hydrogen and iodine to hydrogen iodide.

(ii) Should the constant of proportionality be positive or negative? Why? What are the units of $k$?

(iii) For the purposes of this problem, assume that $k = -1$ and $y(0) = 0.5$. Solve the initial value problem with Euler’s method and discuss the resulting solution in the context of the problem.

(iv) The exact solution to the initial value problem

$$\frac{dy}{dt} = ky^2 \quad \text{where} \quad y(0) = y_0$$

is

$$y(t) = \frac{y_0}{-ky_0t + 1}$$

Using Excel, make a plot of the absolute error and the absolute percent error between the Euler solution and the actual solution over the course of a 50 time-unit experiment.

Absolute Error($t$) = |Actual($t$) − Euler($t$)|

Absolute Percent Error($t$) = $\frac{|Actual(t) - Euler(t)|}{Actual(t)} \times 100$

Discuss what you see and give a physical interpretation of this error plot.

(d) Population Growth

As discussed in previous sections we can make the unconstrained population growth model

$$\frac{dP}{dt} = rP$$

more realistic by multiplying the right-hand side by a function that approaches zero as the population reaches some carrying capacity. The general mathematical model is

Growth Rate = $r \times$ Population size $\times$ Unused Fraction of Capacity

$$\frac{dP}{dt} = r \times P(t) \times U(P(t))$$

Two choices for the $U(P(T))$ function are listed in Table 7.1 Use Euler’s method to plot several solutions to these two population models with $r = 0.25$ and $P(0) = 1$ for various values of $K$. Compare the two models against each other and give a description of the
similarities and differences between the qualitative behavior of the solutions. Start with $K = 10$ and then vary $K$ up and down. You may need to change the total time to get good plots.
Voting Questions

7.3.1 Using Euler’s method, we set up the difference equation $y_{n+1} = y_n + c\Delta t$ to approximate a differential equation. What is the differential equation?

(a) $y' = cy$
(b) $y' = c$
(c) $y' = y + c$
(d) $y' = y + c\Delta t$
(e) None of the above

7.3.2 We know that $f(2) = -3$ and we use Euler’s method to estimate that $f(2.5) \approx -3.6$, when in reality $f(2.5) = -3.3$. This means that between $x = 2$ and $x = 2.5$,

(a) $f(x) > 0$.
(b) $f'(x) > 0$.
(c) $f''(x) > 0$.
(d) $f'''(x) > 0$.
(e) None of the above

7.3.3 We have used Euler’s method and $\Delta t = 0.5$ to approximate the solution to a differential equation with the difference equation $y_{n+1} = y_n + 0.2$. We know that the function $y = 7$ when $t = 2$. What is our approximate value of $y$ when $t = 3$?

(a) $y(3) \approx 7.2$
(b) $y(3) \approx 7.4$
(c) $y(3) \approx 7.6$
(d) $y(3) \approx 7.8$
(e) None of the above

7.3.4 We have used Euler’s method to approximate the solution to a differential equation with the difference equation $z_{n+1} = 1.2z_n$. We know that the function $z(0) = 3$. What is the approximate value of $z(2)$?

(a) $z(2) \approx 3.6$
(b) $z(2) \approx 4.32$
7.3. EULER’S METHOD

(c) $z(2) \approx 5.184$
(d) Not enough information is given.

7.3.5 We have used Euler’s method and $\Delta t = 0.5$ to approximate the solution to a differential equation with the difference equation $y_{n+1} = y_n + t + 0.2$. We know that the function $y = 7$ when $t = 2$. What is our approximate value of $y$ when $t = 3$?

(a) $y(3) \approx 7.4$
(b) $y(3) \approx 11.4$
(c) $y(3) \approx 11.9$
(d) $y(3) \approx 12.9$
(e) None of the above

7.3.6 We have a differential equation for $\frac{dx}{dt}$, we know that $x(0) = 5$, and we want to know $x(10)$. Using Euler’s method and $\Delta t = 1$ we get the result that $x(10) \approx 25.2$. Next, we use Euler’s method again with $\Delta t = 0.5$ and find that $x(10) \approx 14.7$. Finally we use Euler’s method and $\Delta t = 0.25$, finding that $x(10) \approx 65.7$. What does this mean?

(a) These may all be correct. We need to be told which stepsize to use, otherwise we have no way to know which is the right approximation in this context.
(b) Fewer steps means fewer opportunities for error, so $x(10) \approx 25.2$.
(c) Smaller stepsize means smaller errors, so $x(10) \approx 65.7$.
(d) We have no way of knowing whether any of these estimates is anywhere close to the true value of $x(10)$.
(e) Results like this are impossible: We must have made an error in our calculations.

7.3.7 We have a differential equation for $f'(x)$, we know that $f(12) = 0.833$, and we want to know $f(15)$. Using Euler’s method and $\Delta t = 0.1$ we get the result that $f(15) \approx 0.275$. Next, we use $\Delta t = 0.2$ and find that $f(15) \approx 0.468$. When we use $\Delta t = 0.3$, we get $f(15) \approx 0.464$. Finally, we use $\Delta t = 0.4$ and we get $f(15) \approx 0.462$. What does this mean?

(a) These results appear to be converging to $f(15) \approx 0.46$.
(b) Our best estimate is $f(15) \approx 0.275$.
(c) This data does not allow us to make a good estimate of $f(15)$. 
7.4 Separable differential equations

**Preview Activity 7.4.** In this preview activity, we explore whether certain differential equations are separable or not, and then revisit some key ideas from earlier work in integral calculus.

(a) Which of the following differential equations are separable? If the equation is separable, write the equation in the revised form $g(y)\frac{dy}{dt} = h(t)$.

1. $\frac{dy}{dt} = -3y.$
2. $\frac{dy}{dt} = ty - y.$
3. $\frac{dy}{dt} = t + 1.$
4. $\frac{dy}{dt} = t^2 - y^2.$

(b) Explain why any autonomous differential equation is guaranteed to be separable.

(c) Why do we include the term “$+C$” in the expression

$$\int x\,dx = \frac{x^2}{2} + C.$$?

(d) Suppose we know that a certain function $f$ satisfies the equation

$$\int f'(x)\,dx = \int x\,dx.$$?

What can you conclude about $f$?
Activity 7.7.

Suppose that the population of a town is increases by 3% every year.

(a) Let $P(t)$ be the population of the town in year $t$. Write a differential equation that describes the annual growth rate.

(b) Find the solutions of this differential equation.

(c) If you know that the town’s population in year 0 is 10,000, find the population $P(t)$.

(d) How long does it take for the population to double? This time is called the doubling time.

(e) Working more generally, find the doubling time if the annual growth rate is $k$ times the population,
Activity 7.8.

Suppose that a cup of coffee is initially at a temperature of 105° and is placed in a 75° room. Newton’s law of cooling says that

$$\frac{dT}{dt} = -k(T - 75),$$

where $k$ is a constant of proportionality.

(a) Suppose you measure that the coffee is cooling at one degree per minute at the time the coffee is brought into the room. Determine the value of the constant $k$.

(b) Find all the solutions of this differential equation.

(c) What happens to all the solutions as $t \to \infty$? Explain how this agrees with your intuition.

(d) What is the temperature of the cup of coffee after 20 minutes?

(e) How long does it take for the coffee to cool down to 80°?
Activity 7.9.

Solve each of the following differential equations or initial value problems.

(a) \[ \frac{dy}{dt} - (2 - t)y = 2 - t \]

(b) \[ \frac{1}{t} \frac{dy}{dt} = e^{t^2 - 2y} \]

(c) \[ y' = 2y + 2, \quad y(0) = 2 \]

(d) \[ y' = 2y^2, \quad y(-1) = 2 \]

(e) \[ \frac{dy}{dt} = \frac{-2ty}{t^2 + 1}, \quad y(0) = 4 \]
Voting Questions

7.4.1 Which of the following DE’s is/are separable?
(a) $\frac{dy}{dx} = xy$
(b) $\frac{dy}{dx} = x + y$
(c) $\frac{dy}{dx} = \cos(xy)$
(d) Both (a) and (b)
(e) Both (a) and (c)
(f) All of the above

7.4.2 Which of the following differential equations is not separable?
(a) $y' = 3 \sin x \cos y$
(b) $y' = x^2 + 3y$
(c) $y' = e^{2x+y}$
(d) $y' = 4x + 7$
(e) More than one of the above

7.4.3 Which of the following differential equations is not separable?
(a) $\frac{dx}{dt} = xt^2 - 4x$
(b) $\frac{dx}{dt} = 3x^2 t^3$
(c) $\frac{dx}{dt} = \sin(2xt)$
(d) $\frac{dx}{dt} = t^4 \ln(5x)$

7.4.4 Which of the following differential equations is separable?
(a) $uu' = 2x + u$
(b) $3ux = \sin(u')$
(c) $\frac{2x^3}{6u'x^2 + u} = 1$
(d) $e^{2u'x^2} = e^u^3$

7.4.5 If we separate the variables in the differential equation $3z't = z^2$, what do we get?
(a) $3z^{-2}dz = t^{-1}dt$
7.4.6 If we separate the variables in the differential equation \( y' = 2y + 3 \), what do we get?

(a) \( \frac{dy}{3} = dx \)
(b) \( dy = 2y = 3dx \)
(c) \( \frac{dy}{y} = 5dx \)
(d) \( \frac{dy}{2y+3} = dx \)
(e) This equation cannot be separated.

7.4.7 What is the solution to the differential equation: \( \frac{dy}{dx} = 2xy \).

(a) \( y = e^{x^2} + C \)
(b) \( y = Ce^{x^2} \)
(c) \( y = e^{2x} + C \)
(d) \( y = Ce^{2x} \)

7.4.8 The general solution to the equation \( \frac{dy}{dt} = ty \) is

(a) \( y = \frac{t^2}{2} + C \)
(b) \( y = \sqrt{t^2 + C} \)
(c) \( y = e^{t^2/2} + C \)
(d) \( y = Ce^{t^2/2} \)
(e) Trick question, equation is not separable

7.4.9 The general solution to the equation \( \frac{dR}{dy} + R = 1 \) is

(a) \( R = 1 - \sqrt{\frac{1}{C - y}} \)
(b) \( R = 1 - Ce^y \)
(c) \( R = 1 - Ce^{-y} \)
(d) Trick question, equation is not separable
7.4.10 A plant grows at a rate that is proportional to the square root of its height \( h(t) \) – use \( k \) as the constant of proportionality. If we separate the variables in the differential equation for its growth, what do we get?

(a) \( kh^{1/2} dt = dh \)
(b) \( \sqrt{h} dh = kdt \)
(c) \( h^{1/2} dh = kdt \)
(d) \( h^{-1/2} dh = kdt \)
(e) None of the above
7.5 Modeling with differential equations

Preview Activity 7.5. Any time that the rate of change of a quantity is related to the amount of a quantity, a differential equation naturally arises. In the following two problems, we see two such scenarios; for each, we want to develop a differential equation whose solution is the quantity of interest.

(a) Suppose you have a bank account in which money grows at an annual rate of 3%.
   
   (i) If you have $10,000 in the account, at what rate is your money growing?
   
   (ii) Suppose that you are also withdrawing money from the account at $1,000 per year. What is the rate of change in the amount of money in the account? What are the units on this rate of change?

(b) Suppose that a water tank holds 100 gallons and that a salty solution, which contains 20 grams of salt in every gallon, enters the tank at 2 gallons per minute.

   (i) How much salt enters the tank each minute?
   
   (ii) Suppose that initially there are 300 grams of salt in the tank. How much salt is in each gallon at this point in time?
   
   (iii) Finally, suppose that evenly mixed solution is pumped out of the tank at the rate of 2 gallons per minute. How much salt leaves the tank each minute?
   
   (iv) What is the total rate of change in the amount of salt in the tank?
Activity 7.10.
Suppose you have a bank account that grows by 5% every year.

(a) Let $A(t)$ be the amount of money in the account in year $t$. What is the rate of change of $A$?

(b) Suppose that you are also withdrawing $10,000 per year. Write a differential equation that expresses the total rate of change of $A$.

(c) Sketch a slope field for this differential equation, find any equilibrium solutions, and identify them as either stable or unstable. Write a sentence or two that describes the significance of the stability of the equilibrium solution.

(d) Suppose that you initially deposit $100,000 into the account. How long does it take for you to deplete the account?

(e) What is the smallest amount of money you would need to have in the account to guarantee that you never deplete the money in the account?

(f) If your initial deposit is $300,000, how much could you withdraw every year without depleting the account?
Activity 7.11.

A dose of morphine is absorbed from the bloodstream of a patient at a rate proportional to the amount in the bloodstream.

(a) Write a differential equation for \( M(t) \), the amount of morphine in the patient’s bloodstream, using \( k \) as the constant proportionality.

(b) Assuming that the initial dose of morphine is \( M_0 \), solve the initial value problem to find \( M(t) \). Use the fact that the half-life for the absorption of morphine is two hours to find the constant \( k \).

(c) Suppose that a patient is given morphine intravenously at the rate of 3 milligrams per hour. Write a differential equation that combines the intravenous administration of morphine with the body’s natural absorption.

(d) Find any equilibrium solutions and determine their stability.

(e) Assuming that there is initially no morphine in the patient’s bloodstream, solve the initial value problem to determine \( M(t) \).

(f) What happens to \( M(t) \) after a very long time?

(g) Suppose that a doctor asks you to reduce the intravenous rate so that there is eventually 7 milligrams of morphine in the patient’s bloodstream. To what rate would you reduce the intravenous flow?
Voting Questions

7.5.1
7.6 Population Growth and the Logistic Equation

Preview Activity 7.6. Recall that one model for population growth states that a population grows at a rate proportional to its size.

(a) We begin with the differential equation

\[
\frac{dP}{dt} = \frac{1}{2} P.
\]

Sketch a slope field below as well as a few typical solutions on the axes provided.

(b) Find all equilibrium solutions of the equation \( \frac{dP}{dt} = \frac{1}{2} P \) and classify them as stable or unstable.

(c) If \( P(0) \) is positive, describe the long-term behavior of the solution to \( \frac{dP}{dt} = \frac{1}{2} P \).

(d) Let’s now consider a modified differential equation given by

\[
\frac{dP}{dt} = \frac{1}{2} P(3 - P).
\]

As before, sketch a slope field as well as a few typical solutions on the following axes provided.
(e) Find any equilibrium solutions and classify them as stable or unstable.

(f) If \( P(0) \) is positive, describe the long-term behavior of the solution.
Activity 7.12.

Our first model will be based on the following assumption:

_The rate of change of the population is proportional to the population._

On the face of it, this seems pretty reasonable. When there is a relatively small number of people, there will be fewer births and deaths so the rate of change will be small. When there is a larger number of people, there will be more births and deaths so we expect a larger rate of change.

If \( P(t) \) is the population \( t \) years after the year 2000, we may express this assumption as

\[
\frac{dP}{dt} = kP
\]

where \( k \) is a constant of proportionality.

(a) Use the data in the table to estimate the derivative \( P'(0) \) using a central difference. Assume that \( t = 0 \) corresponds to the year 2000.

(b) What is the population \( P(0) \)?

(c) Use these two facts to estimate the constant of proportionality \( k \) in the differential equation.

(d) Now that we know the value of \( k \), we have the initial value problem

\[
\frac{dP}{dt} = kP, \ P(0) = 6.084.
\]

Find the solution to this initial value problem.

(e) What does your solution predict for the population in the year 2010? Is this close to the actual population given in the table?

(f) When does your solution predict that the population will reach 12 billion?

(g) What does your solution predict for the population in the year 2500?

(h) Do you think this is a reasonable model for the earth’s population? Why or why not? Explain your thinking using a couple of complete sentences.
Activity 7.13.

Consider the logistic equation

\[ \frac{dP}{dt} = kP(N - P) \]

with the graph of \( \frac{dP}{dt} \) vs. \( P \) shown below.

(a) At what value of \( P \) is the rate of change greatest?

(b) Consider the model for the earth’s population that we created. At what value of \( P \) is the rate of change greatest? How does that compare to the population in recent years?

(c) According to the model we developed, what will the population be in the year 2100?

(d) According to the model we developed, when will the population reach 9 billion?

(e) Now consider the general solution to the general logistic initial value problem that we found, given by

\[ P(t) = \frac{N}{\left( \frac{N-P_0}{P_0} \right) e^{-kNt} + 1}. \]

Verify algebraically that \( P(0) = P_0 \) and that \( \lim_{t\to\infty} P(t) = N. \)
7.6. POPULATION GROWTH AND THE LOGISTIC EQUATION

Voting Questions

7.6.1 A star’s brightness is decreasing at a rate equal to 10% of its current brightness per million years. If $B_0$ is a constant with units of brightness and $t$ is in millions of years, what function could describe the brightness of the star?

(a) $B'(t) = -0.1B(t)$
(b) $B(t) = B_0e^t$
(c) $B(t) = B_0e^{-0.1t}$
(d) $B(t) = B_0e^{0.1t}$
(e) $B(t) = B_0e^{0.9t}$
(f) $B(t) = -0.1B_0t$

7.6.2 A small company grows at a rate proportional to its size, so that $c'(t) = kc(t)$. We set $t = 0$ in 1990 when there were 50 employees. In 2005 there were 250 employees. What equation must we solve in order to find the growth constant $k$?

(a) $50e^{2005k} = 250$
(b) $50e^{15k} = 250$
(c) $250e^{15k} = 50$
(d) $50e^{tk} = 250$
(e) Not enough information is given.

7.6.3 What differential equation is solved by the function $f(x) = 0.4e^{2x}$?

(a) $\frac{df}{dx} = 0.4f$
(b) $\frac{df}{dx} = 2f$
(c) $\frac{df}{dx} = 2f + 0.4$
(d) $\frac{df}{dx} = 0.4f + 2$
(e) None of the above.

7.6.4 Each of the graphs below show solutions of $y' = k_iy$ for a different $k_i$. Rank these constants from smallest to largest.
7.6. POPULATION GROWTH AND THE LOGISTIC EQUATION

(a) $k_b < k_d < k_a < k_c$
(b) $k_d < k_c < k_b < k_a$
(c) $k_c < k_a < k_d < k_b$
(d) $k_a < k_b < k_c < k_d$

7.6.5 The function $f(y)$ solves the differential equation $f' = -0.1 f$ and we know that $f(0) > 0$. This means that:

(a) When $y$ increases by 1, $f$ decreases by exactly 10%.
(b) When $y$ increases by 1, $f$ decreases by a little more than 10%.
(c) When $y$ increases by 1, $f$ decreases by a little less than 10%.
(d) Not enough information is given.

7.6.6 The function $g(z)$ solves the differential equation $\frac{dg}{dz} = 0.03g$. This means that:

(a) $g$ is an increasing function that changes by 3% every time $z$ increases by 1.
(b) $g$ is an increasing function that changes by more than 3% every time $z$ increases by 1.
(c) $g$ is an increasing function that changes by less than 3% every time $z$ increases by 1.
(d) $g$ is a decreasing function that changes by more than 3% every time $z$ increases by 1.
(e) $g$ is a decreasing function that changes by less than 3% every time $z$ increases by 1.
(f) Not enough information is given.
7.6.7 40 grams of a radioactive element with a half-life of 35 days are put into storage. We solve \( y' = -ky \) with \( k = 0.0198 \) to find a function that describes how the amount of this element will decrease over time. Another facility stores 80 grams of the element and we want to derive a similar function. When solving the differential equation, what value of \( k \) should we use?

(a) \( k = 0.0099 \)
(b) \( k = 0.0198 \)
(c) \( k = 0.0396 \)
(d) None of the above

7.6.8 A star’s brightness is decreasing at a rate equal to 10% of its current brightness per million years, so \( B'(t) = -0.1B(t) \), where \( t \) is measured in millions of years. If we want \( t \) to be measured in years, how would the differential equation change?

(a) \( B'(t) = -0.1B(t) \)
(b) \( B'(t) = -10^5B(t) \)
(c) \( B'(t) = -10^{-6}B(t) \)
(d) \( B'(t) = -10^{-7}B(t) \)
(e) None of the above

7.6.9 The solution to which of the following will approach \( +\infty \) as \( x \) becomes very large?

(a) \( y' = -2y, y(0) = 2 \)
(b) \( y' = 0.1y, y(0) = 1 \)
(c) \( y' = 6y, y(0) = 0 \)
(d) \( y' = 3y, y(0) = -3 \)
(e) None of the above

7.6.10 \( y' = -\frac{1}{3}y \) with \( y(0) = 2 \). As \( x \) becomes large, the solution will

(a) diverge to \( +\infty \).
(b) diverge to \( -\infty \).
(c) approach 0 from above.
(d) approach 0 from below.
(e) do none of the above.
7.6.11 Suppose $H$ is the temperature of a hot object placed into a room whose temperature is 70 degrees, and $t$ represents time. Suppose $k$ is a positive number. Which of the following differential equations best corresponds to Newton’s Law of Cooling?

(a) $\frac{dH}{dt} = -kH$
(b) $\frac{dH}{dt} = k(H - 70)$
(c) $\frac{dH}{dt} = -k(H - 70)$
(d) $\frac{dH}{dt} = -k(70 - H)$
(e) $\frac{dH}{dt} = -kH(H - 70)$

7.6.12 Suppose $H$ is the temperature of a hot object placed into a room whose temperature is 70 degrees. The function $H$ giving the object’s temperature as a function of time is most likely

(a) Increasing, concave up
(b) Increasing, concave down
(c) Decreasing, concave up
(d) Decreasing, concave down

7.6.13 Suppose $H$ is the temperature of a hot object placed into a room whose temperature is 70 degrees, and $t$ represents time. Then $\lim_{t \to \infty} H$ should equal approximately

(a) $-\infty$
(b) 0
(c) 32
(d) 70
(e) Whatever the difference is between the object’s initial temperature and 70

7.6.14 Consider the function $P = \frac{L}{1 + Ae^{-kt}}$, where $A = (L - P_0)/P_0$. Suppose that $P_0 = 10$, $L = 50$, and $k = 0.05$, which of the following could be a graph of this function?
7.6.15 The following graphs all plot the function \( P = \frac{L}{1 + Ae^{-kt}} \). The function plotted in which graph has the largest value of \( L \)?

![Graphs a, b, c, d](image)

7.6.16 The following graphs all plot the function \( P = \frac{L}{1 + Ae^{-kt}} \). The function plotted in which graph has the largest value of \( k \)?

![Graphs a, b, c, d](image)
7.6.17 The following graphs all plot the function \( P = \frac{L}{1 + Ae^{-kt}} \). The function plotted in which graph has the largest value of \( A \)?

7.6.18 Consider the differential equation \( \frac{dP}{dt} = kP \left( 1 - \frac{P}{L} \right) \), called the logistic equation. What are the equilibria of this system?

i. \( k = 0 \) is a stable equilibrium.

ii. \( L = 0 \) is an unstable equilibrium.

iii. \( P = L \) is a stable equilibrium.

iv. \( P = 0 \) is an unstable equilibrium.

v. \( P = L \) is an unstable equilibrium.

vi. \( P = 0 \) is a stable equilibrium.

(a) i
(b) ii
(c) Both iii and v
(d) Both v and vi
(e) Both iv and v
(f) Both iii and iv

7.6.19 The population of rainbow trout in a river system is modeled by the differential equation
\[ \frac{dP}{dt} = 0.2P - 4 \times 10^{-5}P^2. \] What is the maximum number of trout that the river system could support?

(a) \(4 \times 10^5\) trout
(b) 4,000 trout
(c) 5,000 trout
(d) 25,000 trout
(e) Not enough information is given

7.6.20 The solution to the logistic equation
\[ \frac{dP}{dt} = kP \left(1 - \frac{P}{L}\right) \] is
\[ P = \frac{L}{1 + Ae^{-kt}}, \] where \(A = (L - P_0)/P_0\). If we are modeling a herd of elk, with an initial population of 50, in a region with a carrying capacity of 300, and knowing that the exponential growth rate of an elk population is 0.07, which function would describe our elk population as a function of time?

(a) \(P(t) = \frac{300}{1 + 5e^{-0.07t}}\)
(b) \(P(t) = \frac{50}{1 + 5e^{0.07t}}\)
(c) \(P(t) = \frac{300}{1 + 6e^{-0.07t}}\)
(d) \(P(t) = \frac{300}{1 + 6e^{0.07t}}\)

7.6.21 The population of mice on a farm is modeled by the differential equation
\[ \frac{3000}{P} \frac{dP}{dt} = 200 - P. \] If we know that today there are 60 mice on the farm, what function will describe how the mouse population will develop in the future?

(a) \(P = \frac{200}{1 + \frac{1}{4}e^{-t/15}}\)
(b) \(P = \frac{200}{1 + \frac{1}{4}e^{-200t}}\)
(c) \(P = \frac{3000}{1 + 49e^{-t/15}}\)
(d) \(P = \frac{3000}{1 + 49e^{-t/20}}\)
(e) None of the above
7.6.22 The function plotted below could be a solution to which of the following differential equations?

(a) \( \frac{dP}{dt} = -0.05P \left(1 - \frac{P}{80}\right) \)
(b) \( P = \frac{P'}{20 - 6.25 \times 10^{-4} P} \)
(c) \( 40 \frac{P'}{P^2} + \frac{1}{80} = \frac{2}{P} \)
(d) \( -20 \frac{dP}{dt} + P = \frac{P^2}{80} \)
(e) All of the above

7.6.23 The function plotted below could be a solution of which of the following?
(a) $\frac{dP}{dt} = 0.15P \left(1 - \frac{P}{170}\right)$

(b) $\frac{dP}{dt} = 0.15P \left(1 - \frac{P}{240}\right)$

(c) $\frac{dP}{dt} = 0.05P \left(1 - \frac{P}{170}\right)$

(d) $\frac{dP}{dt} = 0.05P \left(1 - \frac{P}{240}\right)$

(e) None of the above
7.7 Linear First Order Differential Equation (Constant Coefficients)

**Preview Activity 7.7.** Suppose a lake has a river running through it. A factory built next to the lake begins releasing a chemical into the lake. The water running in to the lake is free of the chemical, and initially the lake is free of the chemical. The rate at which the amount of chemical in the lake increases \( \frac{dy}{dt} \) is equal to the rate at which the chemical enters the lake minus the rate at which the chemical leaves the lake. Furthermore, the rate at which the chemical leaves the lake is the product of the concentration of the chemical in the lake and the rate at which water leaves the lake.

To get an equation we need some notation. Let \( V \) (m\(^3\)) be the volume of the lake. The phrase “A river running through the lake” means that there is a river running in to the lake at a rate \( R \) m\(^3\)/day and a river running out of the lake also at a rate \( R \) m\(^3\)/day. Assume the factory releases the chemical into the lake at a rate of \( T \) kilograms/day. Finally, let \( y(t) \) be the kilograms of chemical in the lake \( t \) days after the factory begins dumping the chemicals.

(a) Write a differential equation for the pollution in the lake. You can fill in the blanks in the statement of the model to help get started. Hint: be sure to check the units carefully.

\[
\frac{dy}{dt} = \square - \square \cdot \square
\]

(b) What is a reasonable initial condition for the initial value problem associated with the differential equation?

(c) State the equilibrium for the pollution in terms of the parameters \( T, V, \) and \( R, \) and describe the stability of the equilibrium.

(d) Classify the differential equation as linear vs. nonlinear, first order vs second order vs . . . , homogeneous vs. nonhomogeneous.
Activity 7.14.

For each of the following problems, determine if the proposed solution $y(t)$ actually solves the initial value problem.

(a) $y' = -0.4y$ with $y(0) = -7$. The proposed solution is: $y(t) = 7e^{-0.4t}$

(b) $y' + y = 0$ with $y(0) = 5$. The proposed solution is: $y(t) = 5e^t$.

(c) $y' + 2y = 14e^{5t}$ with $y(0) = 1$. The proposed solution is: $y(t) = -e^{-2t} + 2e^{5t}$

(d) $y' + y = 3t + 1$ with $y(0) = -3$. The proposed solution is $y(t) = -e^{-t} + 3t - 2$. 

\[\triangleright\]
Activity 7.15.
Solve each of the linear first order homogeneous initial value problems.

(a) \( y' = -0.4y \) with \( y(0) = -7 \).
(b) \( y' + 4y = 0 \) with \( y(0) = 3 \).
(c) \( 3y' + 7y = 0 \) with \( y(0) = 5 \).
(d) Oil is pumped continuously from a well at a rate proportional to the amount of oil left in the well. Initially there were 1.2 million barrels of oil in the well. Six years later there are 0.3 million barrels. Let \( y(t) \) be the amount of oil in the well. Write a linear first order homogeneous initial value problem describing the situation and solve it.
Activity 7.16.

Solve each of the linear first order nonhomogeneous initial value problems.

(a) \( y' - 0.4y = \sin(\pi t) \) with \( y(0) = -7 \).

(b) \( y' + 4y = 3 \) with \( y(0) = 3 \).

(c) \( 3y' + 7y = 1 - 3t \) with \( y(0) = 5 \).

(d) \( y' - y = 2e^t \) with \( y(0) = -1 \).

(e) \( y' + 2y = \cos(2t) + e^{-t} \) with \( y(0) = 0 \).

(f) Solve the differential equation that arises from Preview Activity 7.7.
Activity 7.17.

For each of the following situations, write a first order nonhomogeneous linear differential equation, find the general solution to the differential equation, and use the given information to find all of the constants and parameters.

(a) A patient is hooked to an IV which infuses 3mg of morphine per hour. The patient’s body absorbs the morphine at a rate proportional to the amount in the bloodstream. Let \( M(t) \) be the amount of morphine in the patient’s bloodstream and let \( k \) be the constant of proportionality. Assume that the patient starts with no morphine in their bloodstream and 1 hour later they have 2mg of morphine in their bloodstream.

(b) When a skydiver jumps from a plane, gravity causes her downward velocity to increase at a rate of \( g \approx -9.81 \) meters per second per second. At the same time, wind resistance causes her downward velocity to decrease at a rate proportional to the velocity. Let \( v(t) \) be the velocity of the skydiver. If there is no initial downward velocity then \( v(0) = 0 \). Assume that 1 second after the jump the sky diver is traveling 8m/s.

(c) Suppose that you have a water tank that holds 100 gallons of water. A briny solution, which contains 20 grams of salt per gallon, enters the take at a rate of 3 gallons per minute. At the same time, the solution is well mixed, and water is pumped out of the tank at the rate of 3 gallons per minute (obviously the water level in the tank remains constant). Let \( S(t) \) denote the number of grams of salt in the tank at minute \( t \). Hint: \( \frac{dS}{dt} = \) (grams per minute going in) - (grams per minute going out). Watch the units carefully.
Voting Questions

7.7.1 Water from a thunderstorm flows into a reservoir at a rate given by the function \( g(t) = 250e^{-0.1t} \), where \( g \) is in gallons per day, and \( t \) is in days. The water in the reservoir evaporates at a rate of 2.25% per day. What equation could describe this scenario?

(a) \( f'(t) = -0.0225f + 250e^{-0.1t} \)
(b) \( f'(t) = -0.0225(250e^{-0.1t}) \)
(c) \( f'(t) = 0.9775f + 250e^{-0.1t} \)
(d) None of the above

7.7.2 The state of ripeness of a banana is described by the differential equation \( R'(t) = 0.05(2 - R) \) with \( R = 0 \) corresponding to a completely green banana and \( R = 1 \) a perfectly ripe banana. If all bananas start completely green, what value of \( R \) describes the state of a completely black, overripe banana?

(a) \( R = 0.05 \)
(b) \( R = \frac{1}{2} \)
(c) \( R = 1 \)
(d) \( R = 2 \)
(e) \( R = 4 \)
(f) None of the above.

7.7.3 The evolution of the temperature \( T \) of a hot cup of coffee cooling off in a room is described by \( \frac{dT}{dt} = -0.01T + 0.6 \), where \( T \) is in °F and \( t \) is in hours. What is the temperature of the room?

(a) 0.6
(b) -0.01
(c) 60
(d) 0.006
(e) 30
(f) none of the above

7.7.4 The evolution of the temperature of a hot cup of coffee cooling off in a room is described by \( \frac{dT}{dt} = -0.01(T - 60) \), where \( T \) is in °F and \( t \) is in hours. Next, we add a small heater to the coffee which adds heat at a rate of 0.1 °F per hour. What happens?
(a) There is no equilibrium, so the coffee gets hotter and hotter.
(b) The coffee reaches an equilibrium temperature of 60°F.
(c) The coffee reaches an equilibrium temperature of 70°F.
(d) The equilibrium temperature becomes unstable.
(e) None of the above

7.7.5 A drug is being administered intravenously into a patient at a certain rate \(d\) and is breaking down at a certain fractional rate \(k > 0\). If \(c(t)\) represents the concentration of the drug in the bloodstream, which differential equation represents this scenario?

(a) \(\frac{dc}{dt} = -k + d\)
(b) \(\frac{dc}{dt} = -kc + d\)
(c) \(\frac{dc}{dt} = kc + d\)
(d) \(\frac{dc}{dt} = c(d - k)\)
(e) None of the above

7.7.6 A drug is being administered intravenously into a patient. The drug is flowing into the bloodstream at a rate of 50 mg/hr. The rate at which the drug breaks down is proportional to the total amount of the drug, and when there is a total of 1000 mg of the drug in the patient, the drug breaks down at a rate of 300 mg/hr. If \(y\) is the number of milligrams of drug in the bloodstream at time \(t\), what differential equation would describe the evolution of the amount of the drug in the patient?

(a) \(y' = -0.3y + 50\)
(b) \(y' = -0.3t + 50\)
(c) \(y' = 0.7y + 50\)
(d) None of the above

7.7.7 The amount of a drug in the bloodstream follows the differential equation \(c' = -kc + d\), where \(d\) is the rate it is being added intravenously and \(k\) is the fractional rate at which it breaks down. If the initial concentration is given by a value \(c(0) > d/k\), then what will happen?

(a) This equation predicts that the concentration of the drug will be negative, which is impossible.
(b) The concentration of the drug will decrease until there is none left.
7.7. LINEAR FIRST ORDER DIFFERENTIAL EQUATION (CONSTANT COEFFICIENTS)

(c) This means that the concentration of the drug will get smaller, until it reaches the level 
\( c = d/k \), where it will stay.

(d) This concentration of the drug will approach but never reach the level \( d/k \).

(e) Because \( c(0) > d/k \) this means that the concentration of the drug will increase, so the 
dose \( d \) should be reduced.

7.7.8 The amount of a drug in the bloodstream follows the differential equation

\[ c' = -kc + d \]

where \( d \) is the rate it is being added intravenously and \( k \) is the fractional rate at which it 
breaks down. If we double the rate at which the drug flows in, how will this change the 
equilibrium value?

(a) It will be double the old value.
(b) It will be greater than the old, but not quite doubled.
(c) It will be more than doubled.
(d) It will be the same.
(e) Not enough information is given.

7.7.9 If we construct an electric circuit with a battery, a resistor, and a capacitor all in series, then 
the voltage is described by the equation \( V_{\text{bat}} = \frac{Q}{C} + IR \). Here \( V_{\text{bat}} \) is the voltage produced 
by the battery, and the constants \( C \) and \( R \) give the capacitance and resistance respectively. \( Q(t) \) 
is the charge on the capacitor and \( I(t) = \frac{dQ}{dt} \) is the current flowing through the circuit. What 
is the equilibrium charge on the capacitor?

(a) \( Q_e = V_{\text{bat}}C \)
(b) \( Q_e = V_{\text{bat}}/R \)
(c) \( Q_e = 0 \)
(d) Not enough information is given.

7.7.10 If we construct an electric circuit with a battery, a resistor, and a capacitor all in series, then 
the voltage is described by the equation \( V_{\text{bat}} = \frac{Q}{C} + IR \). Here \( V_{\text{bat}} \) is the voltage produced 
by the battery, and the constants \( C \) and \( R \) give the capacitance and resistance respectively. \( Q(t) \) 
is the charge on the capacitor and \( I(t) = \frac{dQ}{dt} \) is the current flowing through the circuit. Which of the following functions could describe the charge on the capacitor \( Q(t) \)?

(a) \( Q(t) = 5e^{-t/RC} \)
(b) \( Q(t) = 4e^{-Rt} + V_{\text{bat}}C \)
(c) \( Q(t) = 3e^{-t/RC} - V_{\text{bat}}C \)
(d) \( Q(t) = -6e^{-t/RC} + V_{\text{bat}}C \)
7.7.11 If we construct an electric circuit with a battery, a resistor, and a capacitor all in series, then the voltage is described by the equation $V_{bat} = \frac{Q}{C} + IR$. Here $V_{bat}$ is the voltage produced by the battery, and the constants $C$ and $R$ give the capacitance and resistance respectively. $Q(t)$ is the charge on the capacitor and $I(t) = \frac{dQ}{dt}$ is the current flowing through the circuit. Which of the following functions could describe the current flowing through the circuit $I(t)$?

(a) $I(t) = 5e^{-t/RC}$
(b) $I(t) = 4e^{-RCt} + V_{bat}C$
(c) $I(t) = 3e^{-t/RC} - V_{bat}C$
(d) $I(t) = -6e^{-t/RC} + V_{bat}C$
(e) None of the above
7.8 Linear Second Order Differential Equations (Mass Spring Systems)

Preview Activity 7.8. Consider the mass and spring system in Figure ?? Assuming that the motion is always in the vertical direction, the displacement of the mass at time \( t \) is \( y(t) \), the instantaneous velocity of the mass at the time \( t \) is \( y'(t) \), and the acceleration is \( y''(t) \). Newton's second law: “mass times acceleration equals the sum of the forces” can be used to write

\[
my'' = F_r + F_d + f(t) \tag{7.5}
\]

where \( m \) is the mass of the object, \( F_r \) is the restoring force due to the spring, \( F_d \) is the force due to the damping in the system, and \( f(t) \) represents any external forces on the system.

(a) Hooke’s Law states that the restoring force of the spring is proportional to its displacement. Use the statement of Hooke’s law to propose an expression for the restoring force \( F_r \). The constant of proportionality is called the spring constant. Keep in mind that the restoring force works opposite the displacement so be sure to get the sign correct.

(b) The damping force \( F_d \) is assumed to be proportional to the velocity and acts in the direction opposite the direction of motion. Use this statement to propose an expression for the damping force \( F_d \). The constant of proportionality is called the damping constant. Keep in mind that the damping force works opposite the velocity so be sure to get the sign correct.

(c) If \( f(t) \) is any external force acting on the system then we can finally write a differential equation describing the motion of the mass and spring system. Write this system and give a full description of each of the coefficients.

(d) What are the units of the coefficients given that the units of force are Newtons and

\[
1 \text{ Newton} = \frac{1 \text{ kg} \cdot \text{m}}{\text{s}^2}.
\]
Activity 7.18.

Use the equation derived in Preview Activity 7.8 to change the descriptions of the mass spring systems to a second order linear homogeneous differential equation. Then solve the equation with the aid of Theorem ?? and the given descriptions of the initial displacement and initial velocity. State whether each situation is an over- or under-damped oscillator.

(a) An object with a mass of $m = 1$ kg is suspended from a spring with a spring constant $k = 4$ N/m. The system is submerged in a liquid causing it to have a large damping constant $b = 5$ kg/s. The object is lifted up 1 meter and let go with no initial velocity.

(b) An object with a mass of $m = 1$ kg is suspended from a spring with a spring constant $k = 10$ N/m. The system is submerged in a liquid causing it to have a large damping constant $b = 2$ kg/s. The object is pulled down 1 meter and given an initial velocity of 1 m/s.

(c) An object with a mass of $m = 10$ kg is suspended on a spring with spring constant $k = 20$ N/m. The damping coefficient is $b = 30$ kg/s. The mass is initially held at equilibrium and is given an initial velocity of 2 m/s in the downward direction.
7.8. LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS (MASS SPRING SYSTEMS) 429

Activity 7.19.

Go to the GeoGebra applet
http://www.geogebra.org/student/m217165
This applet is designed to allow you to explore the mass spring system

\[ my'' + by' + ky = f(t) \]

(a) We will start with an un-driven mass spring system where the forcing function is zero. In each of the following cases, sketch a plot of the typical behavior seen.

1. Pick several \( m, b, \) and \( k \) values that generate an over damped system.
2. Pick several \( m, b, \) and \( k \) values that generate a critically damped system.
3. Pick several \( m, b, \) and \( k \) values that generate an under damped system.

(b) Now experiment with a forced spring mass system. Get a feel for what different forcing terms do to control the behavior of the system.

△
7.8.1 A branch sways back and forth with position $f(t)$. Studying its motion you find that its acceleration is proportional to its position, so that when it is 8 cm to the right, it will accelerate to the left at a rate of 2 cm/s$^2$. Which differential equation describes the motion of the branch?

(a) $\frac{d^2f}{dt^2} = 8f$
(b) $\frac{d^2f}{dt^2} = -4f$
(c) $\frac{d^2f}{dt^2} = -2$
(d) $\frac{d^2f}{dt^2} = \frac{f}{4}$
(e) $\frac{d^2f}{dt^2} = -\frac{f}{4}$

7.8.2 The differential equation $\frac{d^2f}{dt^2} = -0.1f + 3$ is solved by a function $f(t)$ where $f$ is in feet and $t$ is in minutes. What units does the number 3 have?

(a) feet
(b) minutes
(c) per minute
(d) per minute$^2$
(e) feet per minute$^2$
(f) no units

7.8.3 The differential equation $y'' = 7y$ is solved by a function $y(t)$ where $y$ is in meters and $t$ is in seconds. What units does the number 7 have?

(a) meters
(b) seconds
(c) per second
(d) per second$^2$
(e) meters per second$^2$
(f) no units

7.8.4 A differential equation is solved by the function $y(t) = 3 \sin 2t$ where $y$ is in meters and $t$ is in seconds. What units do the numbers 3 and 2 have?
7.8. LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS (MASS SPRING SYSTEMS) 431

(a) 3 is in meters, 2 is in seconds
(b) 3 is in meters, 2 is in per second
(c) 3 is in meters per second, 2 has no units
(d) 3 is in meters per second, 2 is in seconds

7.8.5 Three different functions are plotted below. Could these all be solutions of the same second order differential equation?

(a) Yes
(b) No
(c) Not enough information is given.

7.8.6 Which of the following is not a solution of \( y'' + ay = 0 \) for some value of \( a \)?

(a) \( y = 4 \sin 2t \)
(b) \( y = 8 \cos 3t \)
(c) \( y = 2e^{2t} \)
(d) all are solutions

7.8.7 The functions below are solutions of \( y'' + ay = 0 \) for different values of \( a \). Which represents the largest value of \( a \)?

(a) \( y(t) = 100 \sin 2\pi t \)
(b) \( y(t) = 25 \cos 6\pi t \)
(c) \( y(t) = 0.1 \sin 50t \)
(d) \( y(t) = 3 \sin 2t + 8 \cos 2t \)

7.8.8 Each of the differential equations below represents the motion of a mass on a spring. If the mass is the same in each case, which spring is stiffer?

(a) \( s'' + 8s = 0 \)

(b) \( s'' + 2s = 0 \)

(c) \( 2s'' + s = 0 \)

(d) \( 8s'' + s = 0 \)

7.8.9 The motion of a mass on a spring follows the equation \( mx'' = -kx \) where the displacement of the mass is given by \( x(t) \). Which of the following would result in the highest frequency motion?

(a) \( k = 6, m = 2 \)

(b) \( k = 4, m = 4 \)

(c) \( k = 2, m = 6 \)

(d) \( k = 8, m = 6 \)

(e) All frequencies are equal

7.8.10 Each of the differential equations below represents the motion of a mass on a spring. Which system has the largest maximum velocity?

(a) \( 2s'' + 8s = 0, s(0) = 5, s'(0) = 0 \)

(b) \( 2s'' + 4s = 0, s(0) = 7, s'(0) = 0 \)

(c) \( s'' + 4s = 0, s(0) = 10, s'(0) = 0 \)

(d) \( 8s'' + s = 0, s(0) = 20, s'(0) = 0 \)

7.8.11 Which of the following is not a solution of \( \frac{d^2y}{dx^2} = -ay \) for some positive value of \( a \)?

(a) \( y = 2 \sin 6t \)

(b) \( y = 4 \cos 5t \)

(c) \( y = 3 \sin 2t + 8 \cos 2t \)

(d) \( y = 2 \sin 3t + 2 \cos 5t \)
7.8.12 Which function does not solve both $z' = 3z$ and $z'' = 9z$?

(a) $z = 7e^{3t}$
(b) $z = 0$
(c) $z = 12e^{-3t}$
(d) $z = -6e^{3t}$
(e) all are solutions to both

7.8.13 How are the solutions of $y'' = \frac{1}{4}y$ different from solutions of $y' = \frac{1}{2}y$?

(a) The solutions of $y'' = \frac{1}{4}y$ grow half as fast as solutions of $y' = \frac{1}{2}y$.
(b) The solutions of $y'' = \frac{1}{4}y$ include decaying exponentials.
(c) The solutions of $y'' = \frac{1}{4}y$ include sines and cosines.
(d) None of the above

7.8.14 How are the solutions of $y'' = -\frac{1}{4}y$ different from solutions of $y'' = -\frac{1}{2}y$?

(a) The solutions of $y'' = -\frac{1}{4}y$ oscillate twice as fast as the solutions of $y'' = -\frac{1}{2}y$.
(b) The solutions of $y'' = -\frac{1}{4}y$ have a period which is twice as long as the solutions of $y'' = -\frac{1}{2}y$.
(c) The solutions of $y'' = -\frac{1}{4}y$ have a smaller maximum value than the solutions of $y'' = -\frac{1}{2}y$.
(d) More than one of the above is true.
(e) None of the above are true.

7.8.15 What function solves the equation $y'' + 10y = 0$?

(a) $y = 10 \sin 10t$
(b) $y = 60 \cos \sqrt{10}t$
(c) $y = \sqrt{10}e^{-10t}$
(d) $y = 20e\sqrt{10}t$
(e) More than one of the above

7.8.16 We know that the solution of a differential equation is of the form $y = A \sin 3x + B \cos 3x$. Which of the following would tell us that $A = 0$?
434  7.8. LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS (MASS SPRING SYSTEMS)

(a) \( y(0) = 0 \)
(b) \( y'(0) = 0 \)
(c) \( y(1) = 0 \)
(d) none of the above

7.8.17 We know that the solutions to a differential equation are of the form \( y = Ae^{3x} + Be^{-3x} \). If we know that \( y = 0 \) when \( x = 0 \), this means that

(a) \( A = 0 \)
(b) \( B = 0 \)
(c) \( A = -B \)
(d) \( A = B \)
(e) none of the above

7.8.18 An ideal spring produces an acceleration that is proportional to the displacement, so \( my'' = -ky \) for some positive constant \( k \). In the lab, we find that a mass is held on an imperfect spring: As the mass gets farther from equilibrium, the spring produces a force stronger than an ideal spring. Which of the following equations could model this scenario?

(a) \( my'' = ky^2 \)
(b) \( my'' = -k\sqrt{y} \)
(c) \( my'' = -k|y| \)
(d) \( my'' = -ky^3 \)
(e) \( my'' = -ke^{-y} \)
(f) None of the above

7.8.19 The functions plotted below are solutions of \( y'' = -ay \) for different positive values of \( a \). Which case corresponds to the largest value of \( a \)?

[Image of functions]
7.8.20 The motion of a child bouncing on a trampoline is modeled by the equation \( p''(t) + 3p(t) = 6 \) where \( p \) is in inches and \( t \) is in seconds. Suppose we want the position function to be in feet instead of inches. How does this change the differential equation?

(a) There is no change
(b) \( p''(t) + 3p(t) = 0.5 \)
(c) \( p''(t) + 3p(t) = 72 \)
(d) \( 144p''(t) + 3p(t) = 3 \)
(e) \( p''(t) + 36p(t) = 3 \)
(f) \( 144p''(t) + 36p(t) = 3 \)

7.8.21 A float is bobbing up and down on a lake, and the distance of the float from the lake floor follows the equation \( 2d'' + 5d - 30 = 0 \), where \( d(t) \) is in feet and \( t \) is in seconds. At what distance from the lake floor could the float reach equilibrium?

(a) 2 feet
(b) 5 feet  
(c) 30 feet  
(d) 6 feet  
(e) 15 feet  
(f) No equilibrium exists.
7.9 Forced Oscillations

Preview Activity 7.9. For a nonhomogeneous linear differential equation, the general solution takes the form

\[ y(t) = y_h(t) + y_p(t) \]

where \( y_h(t) \) is the homogeneous solution and \( y_p(t) \) is the particular solution given the nonhomogeneity (see Section 7.7). For each of the following second order linear nonhomogeneous differential equations, write the homogeneous solution (see Section 7.8) and a possible particular solution (see Section 7.7).

(a) \( y'' + 5y' + 6y = \sin(2t) \)

(b) \( y'' + 4y = e^{-t} \)

(c) \( y'' + 6y' + 9y = 2 + t \)
Activity 7.20.

Consider differential equation $y'' + 4y = \sin(2t)$. This can be viewed as a mass spring system with a restoring force of $k = 4$, no damping force $b = 0$ and a forcing term $f(t) = \sin(2t)$.

(a) Use the ideas from Section 7.8 to write a general solution to the homogeneous equation $y'' + 4y = 0$.

(b) Conjecture the form of the particular solution $y_p(t)$ that matches the form of the non-homogeneity. In this case the homogeneous solution and the particular solution have exactly the same form. The fix for this is to multiply the particular solution by $t$. Write the particular solution.

(c) Write the solution as the sum of the homogeneous and particular solutions $y(t) = y_h(t) + y_p(t)$.

(d) Use the initial conditions $y(0) = 0$ and $y'(0) = 0$ and the differential equation to find all of the coefficients. State how these initial conditions relate to the mass spring system.

(e) Plot the solution for $0 < t \leq 10$ and explain the behavior you see in relation to the mass spring system.

(f) If the differential equation were changed to $y'' + 3y = \sin(2t)$ (same forcing term but different spring constant), what would you expect from the behavior of the model?
Chapter 8

Sequences and Series

8.1 Sequences

Preview Activity 8.1. Suppose you receive $5000 through an inheritance. You decide to invest this money into a fund that pays 8% annually, compounded monthly. That means that each month your investment earns $\frac{0.08}{12} \cdot P$ additional dollars, where $P$ is your principal balance at the start of the month. So in the first month your investment earns

$$5000 \left( \frac{0.08}{12} \right)$$

or $33.33. If you reinvest this money, you will then have $5033.33 in your account at the end of the first month. From this point on, assume that you reinvest all of the interest you earn.

(a) How much interest will you earn in the second month? How much money will you have in your account at the end of the second month?

(b) Complete Table 8.1 to determine the interest earned and total amount of money in this investment each month for one year.

(c) As we will see later, the amount of money $P_n$ in the account after month $n$ is given by

$$P_n = 5000 \left( 1 + \frac{0.08}{12} \right)^n.$$

Use this formula to check your calculations in Table 8.1. Then find the amount of money in the account after 5 years.

(d) How many years will it be before the account has doubled in value to $10000?
Month | Interest earned | Total amount of money in the account
--- | --- | ---
0 | $0 | $5000.00
1 | $33.33 | $5033.33
2 | 
3 | 
4 | 
5 | 
6 | 
7 | 
8 | 
9 | 
10 | 
11 | 
12 | 

Table 8.1: Interest

Activity 8.1.

(a) Let $s_n$ be the $n$th term in the sequence 1, 2, 3, ... .

Find a formula for $s_n$ and use appropriate technological tools to draw a graph of entries in this sequence by plotting points of the form $(n, s_n)$ for some values of $n$. Most graphing calculators can plot sequences; directions follow for the TI-84.

- In the MODE menu, highlight SEQ in the FUNC line and press ENTER.
- In the Y= menu, you will now see lines to enter sequences. Enter a value for $n$Min (where the sequence starts), a function for $u(n)$ (the $n$th term in the sequence), and the value of $u_{nMin}$.
- Set your window coordinates (this involves choosing limits for $n$ as well as the window coordinates XMin, XMax, YMin, and YMax.
- The GRAPH key will draw a plot of your sequence.

Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.

(b) Let $s_n$ be the $n$th term in the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$. Find a formula for $s_n$. Draw a graph of some points in this sequence. Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.
(c) Let $s_n$ be the $n$th term in the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$. Find a formula for $s_n$. Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence \{ $s_n$ \} has a limit as $n \to \infty$. Explain your reasoning.
Activity 8.2.
(a) Recall our earlier work with limits involving infinity in Section ??.
State clearly what it means for a continuous function \( f \) to have a limit \( L \) as \( x \to \infty \).

(b) Given that an infinite sequence of real numbers is a function from the integers to the real numbers, apply the idea from part (a) to explain what you think it means for a sequence \( \{s_n\} \) to have a limit as \( n \to \infty \).

(c) Based on your response to (b), decide if the sequence \( \left\{ \frac{1+n}{2+n} \right\} \) has a limit as \( n \to \infty \). If so, what is the limit? If not, why not?
Activity 8.3.

Use graphical and/or algebraic methods to determine whether each of the following sequences converges or diverges.

(a) \( \left\{ \frac{1+2n}{3n-2} \right\} \)

(b) \( \left\{ \frac{5+3^n}{10+2^n} \right\} \)

(c) \( \left\{ \frac{10^n}{n!} \right\} \) (where \( ! \) is the factorial symbol and \( n! = n(n - 1)(n - 2) \cdots (2)(1) \) for any positive integer \( n \) (as convention we define 0! to be 1)).
8.1 Sequences

Voting Questions

8.1.1 The sequence \( s_n = \frac{5n + 1}{n} \)

(a) Converges, and the limit is 1
(b) Converges, and the limit is 5
(c) Converges, and the limit is 6
(d) Diverges

8.1.2 The sequence \( s_n = (-1)^n \)

(a) Converges, and the limit is 1
(b) Converges, and the limit is -1
(c) Converges, and the limit is 0
(d) Diverges

8.1.3 The sequence 2, 2.1, 2.11, 2.111, 2.1111, ···

(a) Converges
(b) Diverges

8.1.4 A sequence that is not bounded

(a) Must converge
(b) Might converge
(c) Must diverge
8.2 Geometric Series

**Preview Activity 8.2.** Warfarin is an anticoagulant that prevents blood clotting; often it is prescribed to stroke victims in order to help ensure blood flow. The level of warfarin has to reach a certain concentration in the blood in order to be effective.

Suppose warfarin is taken by a particular patient in a 5 mg dose each day. The drug is absorbed by the body and some is excreted from the system between doses. Assume that at the end of a 24 hour period, $8\%$ of the drug remains in the body. Let $Q_n$ be the amount (in mg) of warfarin in the body before the $(n + 1)$st dose of the drug is administered.

(a) Explain why $Q_1 = 5 \times 0.08$ mg.

(b) Explain why $Q_2 = (5 + Q_1) \times 0.08$ mg. Then show that

$$Q_2 = (5 \times 0.08) (1 + 0.08) \text{ mg.}$$

(c) Explain why $Q_3 = (5 + Q_2) \times 0.08$ mg. Then show that

$$Q_3 = (5 \times 0.08) (1 + 0.08 + 0.08^2) \text{ mg.}$$

(d) Explain why $Q_4 = (5 + Q_3) \times 0.08$ mg. Then show that

$$Q_4 = (5 \times 0.08) (1 + 0.08 + 0.08^2 + 0.08^3) \text{ mg.}$$

(e) There is a pattern that you should see emerging. Use this pattern to find a formula for $Q_n,$ where $n$ is an arbitrary positive integer.

(f) Complete Table 8.2 with values of $Q_n$ for the provided $n$-values (reporting $Q_n$ to 10 decimal places). What appears to be happening to the sequence $Q_n$ as $n$ increases?
Activity 8.4.

Let $a$ and $r$ be real numbers (with $r \neq 0$) and let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

In this activity we will find a shortcut formula for $S_n$ that does not involve a sum of $n$ terms.

(a) Multiply $S_n$ by $r$. What does the resulting sum look like?

(b) Subtract $rS_n$ from $S_n$ and explain why

$$S_n - rS_n = a - ar^n. \quad (8.1)$$

(c) Solve equation (8.1) for $S_n$ to find a simple formula for $S_n$ that does not involve adding $n$ terms.
Activity 8.5.

Let $r > 0$ and $a$ be real numbers and let

$$S = a + ar + ar^2 + \cdots ar^{n-1} + \cdots$$

be an infinite geometric series. For each positive integer $n$, let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$ 

Recall that

$$S_n = a \frac{1 - r^n}{1 - r}.$$ 

(a) What should we allow $n$ to approach in order to have $S_n$ approach $S$?

(b) What is the value of $\lim_{n \to \infty} r^n$ for

- $|r| > 1$?
- $|r| < 1$?

Explain.

(c) If $|r| < 1$, use the formula for $S_n$ and your observations in (a) and (b) to explain why $S$ is finite and find a resulting formula for $S$. 

\(\triangleright\)
Activity 8.6.

The formulas we have derived for the geometric series and its partial sum so far have assumed we begin indexing our sums at \( n = 0 \). If instead we have a sum that does not begin at \( n = 0 \), we can factor out common terms and use our established formulas. This process is illustrated in the examples in this activity.

(a) Consider the sum

\[
\sum_{k=1}^{\infty} \left( 2 \right) \left( \frac{1}{3} \right)^k = \left( 2 \right) \left( \frac{1}{3} \right) + \left( 2 \right) \left( \frac{1}{3} \right)^2 + \left( 2 \right) \left( \frac{1}{3} \right)^3 + \cdots.
\]

Remove the common factor of \( 2 \left( \frac{1}{3} \right) \) from each term and hence find the sum of the series.

(b) Next let \( a \) and \( r \) be real numbers with \(-1 < r < 1\). Consider the sum

\[
\sum_{k=3}^{\infty} ar^k = ar^3 + ar^4 + ar^5 + \cdots.
\]

Remove the common factor of \( ar^3 \) from each term and find the sum of the series.

(c) Finally, we consider the most general case. Let \( a \) and \( r \) be real numbers with \(-1 < r < 1\), let \( n \) be a positive integer, and consider the sum

\[
\sum_{k=n}^{\infty} ar^k = ar^n + ar^{n+1} + ar^{n+2} + \cdots.
\]

Remove the common factor of \( ar^n \) from each term to find the sum of the series.
8.2. GEOMETRIC SERIES

Voting Questions

8.2.1 What will we get if we add up the infinite series of numbers: $16 + 8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$?

(a) This infinite sum will reach a number less than 32.
(b) This infinite sum is equal to 32.
(c) This infinite sum will reach a number greater than 32.
(d) Because we’re adding up an infinite number of numbers which are all greater than zero, the sum diverges to infinity.

8.2.2 What will we get if we add up the infinite series of numbers: $12 + 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots$?

(a) This infinite sum will converge to a number less than 18.
(b) This infinite sum is equal to 18.
(c) This infinite sum will converge a number between 18 and 19.
(d) This infinite sum will converge a number greater than 19.
(e) This infinite sum diverges to infinity.

8.2.3 What will we get if we add up the infinite series of numbers: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots$?

(a) This infinite sum will converge to $\frac{1}{2}$.
(b) This infinite sum will converge to $\frac{2}{3}$.
(c) This infinite sum will converge to 2.
(d) This is not a geometric series.

8.2.4 What will we get if we add up the first 10 terms in the series: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots$?

(a) 0.663
(b) 0.664
(c) 0.666
(d) 0.667
(e) 0.668

8.2.5 What is $\sum_{j=1}^{5} 4j$?

(a) 15
8.2.6 What will we get if we add up the infinite series: \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots\) ?

(a) 2
(b) A number between 2 and 3.
(c) A number between 3 and 4.
(d) A number between 4 and 5.
(e) A number between 5 and 10.
(f) This infinite series diverges to infinity.

8.2.7 Which of the following series is not geometric?

(a) \(\sum_{n=0}^{\infty} \frac{15}{3^n}\)
(b) \(\sum_{n=5}^{\infty} 12^{2n+4}\)
(c) \(\sum_{n=1}^{\infty} 9^{-n}\)
(d) \(\sum_{n=1}^{\infty} 4^{1/n}\)
(e) \(\sum_{n=0}^{\infty} \frac{53^n}{7^{3n}}\)
(f) More than one of these is not geometric.

8.2.8 Which of the following geometric series converge?

(a) \(\sum_{n=0}^{\infty} \frac{8}{(-2)^n}\)
(b) \(\sum_{n=5}^{\infty} 6^{3n+2}\)
(c) \(\sum_{n=1}^{\infty} (-4)^{-n}\)
(d) \(\sum_{n=0}^{\infty} \frac{6^{2n}}{10^{3n}}\)
(e) Exactly two of these converge.
(f) Exactly three of these converge.

8.2.9 Which of the following is/are geometric series?

(a) \(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\)
(b) \(2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \cdots\)
8.2. GEOMETRIC SERIES

(c) $3 + 6 + 12 + 24 + \cdots$
(d) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$
(e) (a) and (b) only
(f) (a), (b), and (c) only
(g) All of the above

8.2.10 $-6 + 4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} =$
(a) $\frac{266}{81}$
(b) $\frac{422}{27}$
(c) $\frac{110}{27}$
(d) $\frac{110}{27}$

8.2.11 $-6 + 4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \cdots$
(a) The sum exists and equals $-18$
(b) The sum exists and equals $-18/5$
(c) The sum exists and equals $18/5$
(d) The sum exists and equals $18$
(e) The sum does not exist

8.2.12 What happens in an infinite geometric series if the common ratio equals 1?
(a) The series has a sum, and it’s equal to 1
(b) The series has a sum, but the sum depends on the first term
(c) The series does not have a sum because the partial sums don’t have a limit


8.3 Series of Real Numbers

Preview Activity 8.3. Have you ever wondered how your calculator can produce a numeric approximation for complicated numbers like \( e, \pi \) or \( \ln(2) \)? After all, the only operations a calculator can really perform are addition, subtraction, multiplication, and division, the operations that make up polynomials. This activity provides the first steps in understanding how this process works. Throughout the activity, let \( f(x) = e^x \).

(a) Find the tangent line to \( f \) at \( x = 0 \) and use this linearization to approximate \( e \). That is, find a formula \( L(x) \) for the tangent line, and compute \( L(1) \), since \( L(1) \approx f(1) = e \).

(b) The linearization of \( e^x \) does not provide a good approximation to \( e \) since 1 is not very close to 0. To obtain a better approximation, we alter our approach a bit. Instead of using a straight line to approximate \( e^x \), we put an appropriate bend in our estimating function to make it better fit the graph of \( e^x \) for \( x \) close to 0. With the linearization, we had both \( f(x) \) and \( f'(x) \) share the same value as the linearization at \( x = 0 \). We will now use a quadratic approximation \( P_2(x) \) to \( f(x) = e^x \) centered at \( x = 0 \) which has the property that \( P_2(0) = f(0) \), \( P_2'(0) = f'(0) \), and \( P_2''(0) = f''(0) \).

(i) Let \( P_2(x) = 1 + x + \frac{x^2}{2} \). Show that \( P_2(0) = f(0) \), \( P_2'(0) = f'(0) \), and \( P_2''(0) = f''(0) \). Then, use \( P_2(x) \) to approximate \( e \) by observing that \( P_2(1) \approx f(1) \).

(ii) We can continue approximating \( e \) with polynomials of larger degree whose higher derivatives agree with those of \( f \) at 0. This turns out to make the polynomials fit the graph of \( f \) better for more values of \( x \) around 0. For example, let \( P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \). Show that \( P_3(0) = f(0) \), \( P_3'(0) = f'(0) \), \( P_3''(0) = f''(0) \), and \( P_3'''(0) = f'''(0) \). Use \( P_3(x) \) to approximate \( e \) in a way similar to how you did so with \( P_2(x) \) above.

\[ \Box \]
Activity 8.7.
Consider the series
\[ \sum_{k=1}^{\infty} \frac{1}{k^2}. \]
While it is physically impossible to add an infinite collection of numbers, we can, of course, add any finite collection of them. In what follows, we investigate how understanding how to find the \( n \)th partial sum (that is, the sum of the first \( n \) terms) enables us to make sense of what the infinite sum.

(a) Sum the first two numbers in this series. That is, find a numeric value for
\[ \sum_{k=1}^{2} \frac{1}{k^2} \]

(b) Next, add the first three numbers in the series.

(c) Continue adding terms in this series to complete Table 3. Carry each sum to at least 8 decimal places.

\[ \begin{array}{c|c|c}
\sum_{k=1}^{1} \frac{1}{k^2} &= 1 \\
\sum_{k=1}^{2} \frac{1}{k^2} &= 1 \\
\sum_{k=1}^{3} \frac{1}{k^2} &= 3 \\
\sum_{k=1}^{4} \frac{1}{k^2} &= 4 \\
\sum_{k=1}^{5} \frac{1}{k^2} &= 5 \\
\sum_{k=1}^{6} \frac{1}{k^2} &= 6 \\
\sum_{k=1}^{7} \frac{1}{k^2} &= 7 \\
\sum_{k=1}^{8} \frac{1}{k^2} &= 8 \\
\sum_{k=1}^{9} \frac{1}{k^2} &= 9 \\
\sum_{k=1}^{10} \frac{1}{k^2} &= 10 \\
\end{array} \]

Table 8.3: Sums of some of the first terms of the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \)

(d) The sums in the table in (c) form a sequence whose \( n \)th term is \( S_n = \sum_{k=1}^{n} \frac{1}{k^2} \). Based on your calculations in the table, do you think the sequence \( \{S_n\} \) converges or diverges? Explain. How do you think this sequence \( \{S_n\} \) is related to the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \)?

\[ \triangleq \]
Activity 8.8.

If the series $\sum a_k$ converges, then an important result necessarily follows regarding the sequence $\{a_n\}$. This activity explores this result.

Assume that the series
\[ \sum_{k=1}^{\infty} a_k \]
converges and has sum equal to $L$. What does this fact tell us about the sequence $\{a_n\}$?

(a) What is the $n$th partial sum $S_n$ of the series $\sum_{k=1}^{\infty} a_k$?

(b) What is the $(n - 1)$st partial sum $S_{n-1}$ of the series $\sum_{k=1}^{\infty} a_k$?

(c) What is the difference between the $n$th partial sum and the $(n - 1)$st partial sum of the series $\sum_{k=1}^{\infty} a_k$?

(d) Since we are assuming that $\sum_{k=1}^{\infty} a_k = L$, what does that tell us about $\lim_{n \to \infty} S_n$? Why?

What does that tell us about $\lim_{n \to \infty} S_{n-1}$? Why?

(e) Combine the results of the previous two parts of this activity to determine $\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1})$. 

\[ \Box \]
Activity 8.9.

Determine if the Divergence Test applies to the following series. If the test does not apply, explain why. If the test does apply, what does it tell us about the series?

(a) \( \sum \frac{1}{k+1} \)

(b) \( \sum (-1)^k \)

(c) \( \sum \frac{1}{k} \)
Activity 8.10.

Consider the harmonic series \( \sum_{k=1}^{\infty} \frac{1}{k} \). Recall that the harmonic series will converge provided that its sequence of partial sums converges. The \( n \)th partial sum \( S_n \) of the series \( \sum_{k=1}^{\infty} \frac{1}{k} \) is

\[
S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = 1(1) + (1) \left( \frac{1}{2} \right) + (1) \left( \frac{1}{3} \right) + \cdots + (1) \left( \frac{1}{n} \right).
\]

Through this last expression for \( S_n \), we can visualize this partial sum as a sum of areas of rectangles with heights \( \frac{1}{m} \) and bases of length 1, as shown in Figure 8.1, which uses the 9th partial sum. The graph of the continuous function \( f \) defined by \( f(x) = \frac{1}{x} \) is overlaid on this plot.

![Figure 8.1: A picture of the 9th partial sum of the harmonic series as a sum of areas of rectangles.](image-url)

(a) Explain how this picture represents a particular Riemann sum.

(b) What is the definite integral that corresponds to the Riemann sum you considered in (a)?

(c) Which is larger, the definite integral in (b), or the corresponding partial sum \( S_9 \) of the series? Why?

(d) If instead of considering the 9th partial sum, we consider the \( n \)th partial sum, and we let \( n \) go to infinity, we can then compare the series \( \sum_{k=1}^{\infty} \frac{1}{k} \) to the improper integral \( \int_{1}^{\infty} \frac{1}{x} \, dx \).
Which of these quantities is larger? Why?

(e) Does the improper integral \( \int_1^\infty \frac{1}{x} \, dx \) converge or diverge? What does that result, together with your work in (d), tell us about the series \( \sum_{k=1}^\infty \frac{1}{k} \)?
Activity 8.11.

The series \( \sum \frac{1}{k^p} \) are special series called \( p \)-series. We have already seen that the \( p \)-series with \( p = 1 \) (the harmonic series) diverges. We investigate the behavior of other \( p \)-series in this activity.

(a) Evaluate the improper integral \( \int_1^\infty \frac{1}{x^2} \, dx \). Does the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converge or diverge? Explain.

(b) Evaluate the improper integral \( \int_1^\infty \frac{1}{x^p} \, dx \) where \( p > 1 \). For which values of \( p \) can we conclude that the series \( \sum_{k=1}^{\infty} \frac{1}{k^p} \) converges?

(c) Evaluate the improper integral \( \int_1^\infty \frac{1}{x^p} \, dx \) where \( p < 1 \). What does this tell us about the corresponding \( p \)-series \( \sum_{k=1}^{\infty} \frac{1}{k^p} \)?

(d) Summarize your work in this activity by completing the following statement.

The \( p \)-series \( \sum_{k=1}^{\infty} \frac{1}{k^p} \) converges if and only if __________________________.
Activity 8.12.

Consider the series $\sum \frac{k+1}{k^3+2}$. Since the convergence or divergence of a series only depends on the behavior of the series for large values of $k$, we might examine the terms of this series more closely as $k$ gets large.

(a) By computing the value of $\frac{k+1}{k^3+2}$ for $k = 100$ and $k = 1000$, explain why the terms $\frac{k+1}{k^3+2}$ are essentially $\frac{k}{k^3}$ when $k$ is large.

(b) Let’s formalize our observations in (a) a bit more. Let $a_k = \frac{k+1}{k^3+2}$ and $b_k = \frac{k}{k^3}$. Calculate

$$\lim_{k \to \infty} \frac{a_k}{b_k}.$$ 

What does the value of the limit tell you about $a_k$ and $b_k$ for large values of $k$? Compare your response from part (a).

(c) Does the series $\sum \frac{k}{k^3}$ converge or diverge? Why? What do you think that tells us about the convergence or divergence of the series $\sum \frac{k+1}{k^3+2}$? Explain.

\[<\]
Activity 8.13.

Use the Limit Comparison Test to determine the convergence or divergence of the series

\[
\sum \frac{3k^2 + 1}{5k^4 + 2k + 2}
\]

by comparing it to the series \( \sum \frac{1}{k^2} \).

Consider the series defined by

$$\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}.$$  \hfill (8.2)

This series is not a geometric series, but this activity will illustrate how we might compare this series to a geometric one. Recall that a series $\sum a_k$ is geometric if the ratio $\frac{a_{k+1}}{a_k}$ is always the same. For the series in (8.2), note that $a_k = \frac{2^k}{k+3\pi}$.

(a) To see if $\sum \frac{2^k}{3^k - k}$ is comparable to a geometric series, we analyze the ratios of successive terms in the series. Complete Table 1, listing your calculations to at least 8 decimal places.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\frac{a_{k+1}}{a_k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>

Table 8.4: Ratios of successive terms in the series $\sum \frac{2^k}{3^k - k}$

(b) Based on your calculations in Table 1, what can we say about the ratio $\frac{a_{k+1}}{a_k}$ if $k$ is large?

(c) Do you agree or disagree with the statement: “the series $\sum \frac{2^k}{3^k - k}$ is approximately geometric when $k$ is large”? If not, why not? If so, do you think the series $\sum \frac{2^k}{3^k - k}$ converges or diverges? Explain.

$\triangleright$
Activity 8.15.

Determine whether each of the following series converges or diverges. Explicitly state which test you use.

(a) \[ \sum \frac{k}{2^k} \]

(b) \[ \sum \frac{k^3 + 2}{k^2 + 1} \]

(c) \[ \sum \frac{10^k}{k!} \]

(d) \[ \sum \frac{k^3 - 2k^2 + 1}{k^6 + 4} \]
8.3. SERIES OF REAL NUMBERS

Voting Questions

8.3.1 The sum of the series \[ \frac{15}{2} + \frac{45}{8} + \frac{135}{32} + \frac{405}{128} + \frac{1215}{512} + \cdots \]

(a) Exists
(b) Does not exist

8.3.2 The series \( \sum_{n=1}^{\infty} \frac{n}{10} \)

(a) Converges
(b) Diverges

8.3.3 If \( a_n \to 0 \) as \( n \to \infty \), then \( \sum_{n=1}^{\infty} a_n \) converges.

(a) Always true
(b) Not always true

8.3.4 If \( a_n \) is a convergent sequence, then \( \sum_{n=1}^{\infty} a_n \) is a convergent series.

(a) True
(b) False

8.3.5 The series \( \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \)

(a) Converges
(b) Diverges

8.3.6 For what values of \( p \) does the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converge?

(a) This series converges for all values of \( p \).
(b) This series converges only if \( p > 2 \).
(c) This series converges only if \( p > 1 \).
(d) This series converges only if $p > 0$.

(e) This series does not converge for any values of $p$.

8.3.7 The series $\sum_{n=1}^{\infty} \left( \frac{10}{n^5} + \frac{(-3)^n}{4^n} \right)$

(a) Converges

(b) Diverges

8.3.8 The series $\sum_{n=1}^{\infty} \left( \frac{1}{2^n} + \frac{1}{n} \right)$

(a) Converges

(b) Diverges

8.3.9 The series $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)}$

(a) Converges

(b) Diverges

8.3.10 The series $\sum_{n=1}^{\infty} \left( \frac{1}{2^n} + \frac{1}{n} \right)$

(a) Converges

(b) Diverges

8.3.11 Does the series $\sum_{n=1}^{\infty} \frac{100}{n^2 + 2}$ converge?

(a) Yes, this series converges.

(b) No, this series does not converge.

(c) It is impossible to tell.

8.3.12 If $a_n > b_n$ for all $n$ and $\sum b_n$ converges, then

(a) $\sum a_n$ converges

(b) $\sum a_n$ diverges
(c) Not enough information to determine convergence or divergence of \( \sum a_n \)

8.3.13 The best way to test the series \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \) for convergence or divergence is

(a) Looking at the sequence of partial sums  
(b) Using rules for geometric series  
(c) The Integral Test  
(d) Using rules for \( p \)-series  
(e) The Comparison Test  
(f) The Limit Comparison Test

8.3.14 Does the series \( \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \) converge?

(a) This series converges.  
(b) This series diverges.  
(c) It is impossible to tell.

8.3.15 The series \( \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2 + 1} \)

(a) Converges  
(b) Diverges

8.3.16 The series \( \sum_{n=1}^{\infty} (n^{-1.4} + 3n^{-1.2}) \)

(a) Converges  
(b) Diverges

8.3.17 The series \( \sum_{n=1}^{\infty} \frac{1}{ne^n} \)

(a) Converges  
(b) Diverges
8.3.18 The series \( \sum_{n=1}^{\infty} \frac{(n - 1)!}{5^n} \)

(a) Converges
(b) Diverges

8.3.19 Does the series \( \sum_{n=1}^{\infty} \frac{n^3}{3^n} \) converge?

(a) This series converges.
(b) This series diverges.
(c) It is impossible to tell.

8.3.20 Does the series \( \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \) converge?

(a) This series converges.
(b) This series diverges.
(c) It is impossible to tell.

8.3.21 Does the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converge?

(a) This series converges.
(b) This series diverges.
(c) It is impossible to tell.
8.4 Alternating Series

**Preview Activity 8.4.** Preview Activity 8.3 showed how we can approximate the number $e$ with linear, quadratic, and other polynomial approximations. We use a similar approach in this activity to obtain linear and quadratic approximations to $\ln(2)$. Along the way, we encounter a type of series that is different than most of the ones we have seen so far. Throughout this activity, let $f(x) = \ln(1 + x)$.

(a) Find the tangent line to $f$ at $x = 0$ and use this linearization to approximate $\ln(2)$. That is, find $L(x)$, the tangent line approximation to $f(x)$, and use the fact that $L(1) \approx f(1)$ to estimate $\ln(2)$.

(b) The linearization of $\ln(1 + x)$ does not provide a very good approximation to $\ln(2)$ since 1 is not that close to 0. To obtain a better approximation, we alter our approach; instead of using a straight line to approximate $\ln(2)$, we use a quadratic function to account for the concavity of $\ln(1 + x)$ for $x$ close to 0. With the linearization, both the function’s value and slope agree with the linearization’s value and slope at $x = 0$. We will now make a quadratic approximation $P_2(x)$ to $f(x) = \ln(1 + x)$ centered at $x = 0$ with the property that $P_2(0) = f(0)$, $P'_2(0) = f'(0)$, and $P''_2(0) = f''(0)$.

(i) Let $P_2(x) = x - \frac{x^2}{2}$. Show that $P_2(0) = f(0)$, $P'_2(0) = f'(0)$, and $P''_2(0) = f''(0)$. Use $P_2(x)$ to approximate $\ln(2)$ by using the fact that $P_2(1) \approx f(1)$.

(ii) We can continue approximating $\ln(2)$ with polynomials of larger degree whose derivatives agree with those of $f$ at 0. This makes the polynomials fit the graph of $f$ better for more values of $x$ around 0. For example, let $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. Show that $P_3(0) = f(0)$, $P'_3(0) = f'(0)$, $P''_3(0) = f''(0)$, and $P'''_3(0) = f'''(0)$. Taking a similar approach to preceding questions, use $P_3(x)$ to approximate $\ln(2)$.

(iii) If we used a degree 4 or degree 5 polynomial to approximate $\ln(1 + x)$, what approximations of $\ln(2)$ do you think would result? Use the preceding questions to conjecture a pattern that holds, and state the degree 4 and degree 5 approximation.
Activity 8.16.

Remember that, by definition, a series converges if and only if its corresponding sequence of partial sums converges.

(a) Complete the following table by calculating the first few partial sums (to 10 decimal places) of the alternating series

\[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \]

<table>
<thead>
<tr>
<th>Partial Sum</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{k=1}^{1} (-1)^{k+1} \frac{1}{k} )</td>
<td>1.0000000000</td>
</tr>
<tr>
<td>( \sum_{k=1}^{2} (-1)^{k+1} \frac{1}{k} )</td>
<td>0.6666666667</td>
</tr>
<tr>
<td>( \sum_{k=1}^{3} (-1)^{k+1} \frac{1}{k} )</td>
<td>0.5000000000</td>
</tr>
<tr>
<td>( \sum_{k=1}^{4} (-1)^{k+1} \frac{1}{k} )</td>
<td>0.3333333333</td>
</tr>
<tr>
<td>( \sum_{k=1}^{5} (-1)^{k+1} \frac{1}{k} )</td>
<td>0.2500000000</td>
</tr>
</tbody>
</table>

Table 8.5: Partial sums of the alternating series \( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \)

(b) Plot the sequence of partial sums from part (a) in the plane. What do you notice about this sequence?
Activity 8.17.

Which series converge and which diverge? Justify your answers.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 + 2}$

(b) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}2k}{k + 5}$

(c) $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$
Activity 8.18.

Determine the number of terms it takes to approximate the sum of the convergent alternating series

\[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \]

to within 0.0001.
Activity 8.19.

(a) Explain why the series
\[
1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots
\]
must have a sum that is less than the series
\[
\sum_{k=1}^{\infty} \frac{1}{k^2}.
\]

(b) Explain why the series
\[
1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots
\]
must have a sum that is greater than the series
\[
\sum_{k=1}^{\infty} -\frac{1}{k^2}.
\]

(c) Given that the terms in the series
\[
1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots
\]
converge to 0, what do you think the previous two results tell us about the convergence status of this series?
Activity 8.20.

(a) Consider the series $\sum (-1)^k \frac{\ln(k)}{k}$.
   (i) Does this series converge? Explain.
   (ii) Does this series converge absolutely? Explain what test you use to determine your answer.

(b) Consider the series $\sum (-1)^k \frac{\ln(k)}{k^2}$.
   (i) Does this series converge? Explain.
   (ii) Does this series converge absolutely? Hint: Use the fact that $\ln(k) < \sqrt{k}$ for large values of $k$ and the compare to an appropriate $p$-series.
Activity 8.21.

For (a)-(j), use appropriate tests to determine the convergence or divergence of the following series. Throughout, if a series is a convergent geometric series, find its sum.

(a) \[ \sum_{k=3}^{\infty} \frac{2}{\sqrt{k} - 2} \]

(b) \[ \sum_{k=1}^{\infty} \frac{k}{1 + 2k} \]

(c) \[ \sum_{k=0}^{\infty} \frac{2k^2 + 1}{k^3 + k + 1} \]

(d) \[ \sum_{k=0}^{\infty} \frac{100^k}{k!} \]

(e) \[ \sum_{k=1}^{\infty} \frac{2^k}{5^k} \]

(f) \[ \sum_{k=1}^{\infty} \frac{k^3 - 1}{k^5 + 1} \]

(g) \[ \sum_{k=2}^{\infty} \frac{3^{k-1}}{7^k} \]

(h) \[ \sum_{k=2}^{\infty} \frac{1}{k^k} \]

(i) \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k} + 1} \]

(j) \[ \sum_{k=2}^{\infty} \frac{1}{k \ln(k)} \]

(k) Determine a value of \( n \) so that the \( n \)th partial sum \( S_n \) of the alternating series \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} \]

approximates the sum to within 0.001.

\[ \triangle \]
Voting Questions

8.4.1
8.5 Taylor Polynomials and Taylor Series

Preview Activity 8.5. Preview Activity 8.3 showed how we can approximate the number $e$ using linear, quadratic, and other polynomial functions; we then used similar ideas in Preview Activity 8.4 to approximate $\ln(2)$. In this activity, we review and extend the process to find the “best” quadratic approximation to the exponential function $e^x$ around the origin. Let $f(x) = e^x$ throughout this activity.

(a) Find a formula for $P_1(x)$, the linearization of $f(x)$ at $x = 0$. (We label this linearization $P_1$ because it is a first degree polynomial approximation.) Recall that $P_1(x)$ is a good approximation to $f(x)$ for values of $x$ close to 0. Plot $f$ and $P_1$ near $x = 0$ to illustrate this fact.

(b) Since $f(x) = e^x$ is not linear, the linear approximation eventually is not a very good one. To obtain better approximations, we want to develop a different approximation that “bends” to make it more closely fit the graph of $f$ near $x = 0$. To do so, we add a quadratic term to $P_1(x)$. In other words, we let

$$P_2(x) = P_1(x) + c_2x^2$$

for some real number $c_2$. We need to determine the value of $c_2$ that makes the graph of $P_2(x)$ best fit the graph of $f(x)$ near $x = 0$.

Remember that $P_1(x)$ was a good linear approximation to $f(x)$ near 0; this is because $P_1(0) = f(0)$ and $P_1'(0) = f'(0)$. It is therefore reasonable to seek a value of $c_2$ so that

$$P_2(0) = f(0),$$
$$P_2'(0) = f'(0),$$
$$P_2''(0) = f''(0).$$

Remember, we are letting $P_2(x) = P_1(x) + c_2x^2$.

(i) Calculate $P_2(0)$ to show that $P_2(0) = f(0)$.

(ii) Calculate $P_2'(0)$ to show that $P_2'(0) = f'(0)$.

(iii) Calculate $P_2''(x)$. Then find a formula for $c_2$ so that $P_2''(0) = f''(0)$.

(iv) Explain why the condition $P_2''(0) = f''(0)$ will put an appropriate “bend” in the graph of $P_2$ to make $P_2$ fit the graph of $f$ around $x = 0$. 

$\Box$
Activity 8.22.

We have just seen that the $n$th order Taylor polynomial centered at $a = 0$ for the exponential function $e^x$ is

$$
\sum_{k=0}^{n} \frac{x^k}{k!}.
$$

In this activity, we determine the $n$th order Taylor polynomials for several other familiar functions.

(a) Let $f(x) = \frac{1}{1-x}$.

(i) Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find a general formula for $f^{(k)}(0)$.

(ii) Determine the $n$th order Taylor polynomial for $f(x) = \frac{1}{1-x}$ centered at $x = 0$.

(b) Let $f(x) = \cos(x)$.

(i) Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find a general formula for $f^{(k)}(0)$. (Think about how $k$ being even or odd affects the value of the $k$th derivative.)

(ii) Determine the $n$th order Taylor polynomial for $f(x) = \cos(x)$ centered at $x = 0$.

(c) Let $f(x) = \sin(x)$.

(i) Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find a general formula for $f^{(k)}(0)$. (Think about how $k$ being even or odd affects the value of the $k$th derivative.)

(ii) Determine the $n$th order Taylor polynomial for $f(x) = \sin(x)$ centered at $x = 0$. Hint: Your answer should depend on whether $n$ is even or odd.
Activity 8.23.

(a) Plot the graphs of several of Taylor polynomials centered at 0 (of order at least 5) for $e^x$ and convince yourself that these Taylor polynomials converge to $e^x$ for every value of $x$.

(b) Draw the graphs of several of the Taylor polynomials centered at 0 (of order at least 6) for $\cos(x)$ and convince yourself that these Taylor polynomials converge to $\cos(x)$ for every value of $x$. Write the Taylor series centered at 0 for $\cos(x)$.

(c) Draw the graphs of several of the Taylor polynomials centered at 0 for $\frac{1}{1-x}$. Based on your graphs, for what values of $x$ do these Taylor polynomials appear converge to $\frac{1}{1-x}$? How is this situation different from what we observe with $e^x$ and $\cos(x)$? In addition, write the Taylor series centered at 0 for $\frac{1}{1-x}$.
Activity 8.24.

(a) Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for \( f(x) = \frac{1}{1-x} \) centered at \( x = 0 \).

(b) Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for \( f(x) = \cos(x) \) centered at \( x = 0 \).

(c) Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for \( f(x) = \sin(x) \) centered at \( x = 0 \).
Activity 8.25.

Let $P_n(x)$ be the $n$th order Taylor polynomial for $\sin(x)$ centered at $x = 0$. Determine how large we need to choose $n$ so that $P_n(2)$ approximates $\sin(2)$ to 20 decimal places.

(a) Show that the Taylor series centered at 0 for $\cos(x)$ converges to $\cos(x)$ for every real number $x$.

(b) Next we consider the Taylor series for $e^x$.
   
   (i) Show that the Taylor series centered at 0 for $e^x$ converges to $e^x$ for every nonnegative value of $x$.
   
   (ii) Show that the Taylor series centered at 0 for $e^x$ converges to $e^x$ for every negative value of $x$.
   
   (iii) Explain why the Taylor series centered at 0 for $e^x$ converges to $e^x$ for every real number $x$. Recall that we earlier showed that the Taylor series centered at 0 for $e^x$ converges for all $x$, and we have now completed the argument that the Taylor series for $e^x$ actually converges to $e^x$ for all $x$.

(c) Let $P_n(x)$ be the $n$th order Taylor polynomial for $e^x$ centered at 0. Find a value of $n$ so that $P_n(5)$ approximates $e^5$ correct to 8 decimal places.

\[ \boxed{} \]
8.5 TAYLOR POLYNOMIALS AND TAYLOR SERIES

Voting Questions

8.5.1 Find the Taylor series for the function $\ln(x)$ at the point $a = 1$. (No calculators allowed.)

(a) $(x - 1) - \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3 - \frac{1}{4} (x - 1)^4 + \cdots$
(b) $x - 1 - (x - 1)^2 + 2(x - 1)^3 - 6(x - 1)^4 + \cdots$
(c) $\ln(x) + \frac{1}{x}(x - 1) - \frac{1}{2x^2}(x - 1)^2 + \frac{2}{3x^3}(x - 1)^3 - \frac{6}{4x^4}(x - 1)^4 + \cdots$
(d) $\ln(x) + \frac{1}{x}(x - 1) - \frac{1}{2x^2}(x - 1)^2 + \frac{1}{3x^3}(x - 1)^3 - \frac{1}{4x^4}(x - 1)^4 + \cdots$
(e) This is not possible.

8.5.2 If $a = 0$, what function is represented by the Taylor series $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$? (No calculators allowed.)

(a) $e^x$
(b) $\sin x$
(c) $\cos x$
(d) This is not a Taylor series.

8.5.3 A Taylor series converges when $x = 12, 13$ and $15$, but diverges when $x = 9, 16$ and $18$. Which of the following could be $a$, the point where the Taylor series is centered?

(a) $a = 9$
(b) $a = 11$
(c) $a = 13$
(d) $a = 15$
(e) All of the above are possible.
(f) None of the above are possible.

8.5.4 Suppose we find a Taylor series for the function $f(x)$ centered at the point $a = 5$. Where would we expect a finite number of terms from this Taylor series to probably give us a better estimate?

(a) $x = 0$
(b) $x = 3$
(c) $x = 8$
(d) There is no way to tell.
8.5.5 A Taylor series for a function \( f(x) \) at \( a = 10 \) has a radius of convergence of 3. If we use the first 10 terms of this series to estimate \( f(15) \) we will probably get

(a) an infinite result.
(b) a result which is closer to the real value of \( f(15) \) than if we used 5 terms.
(c) a result which is farther from the real value of \( f(15) \) than if we used 25 terms.
(d) a result which is closer to the real value of \( f(15) \) than if we used 15 terms.
(e) More than one of the above are true.

8.5.6 We are given a Taylor series for a function \( g(x) \) at \( a = -5 \), with a radius of convergence of 6. Which would give the best estimate of \( g(-5) \)?

(a) The first term of the Taylor series.
(b) The first 5 terms of the Taylor series.
(c) The first 10 terms of the Taylor series.
(d) The first 100 terms of the Taylor series.
(e) All would give the same result.
8.6 Power Series

Preview Activity 8.6. In Chapter 7, we learned some of the many important applications of differential equations, and learned some approaches to solve or analyze them. Here, we consider an important approach that will allow us to solve a wider variety of differential equations.

Let’s consider the familiar differential equation from exponential population growth given by

$$y' = ky,$$  \hspace{1cm} (8.3)

where $k$ is the constant of proportionality. While we can solve this differential equation using methods we have already learned, we take a different approach now that can be applied to a much larger set of differential equations. For the rest of this activity, let’s assume that $k = 1$. We will use our knowledge of Taylor series to find a solution to the differential equation (8.3).

To do so, we assume that we have a solution $y = f(x)$ and that $f(x)$ has a Taylor series that can be written in the form

$$y = f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where the coefficients $a_k$ are undetermined. Our task is to find the coefficients.

(a) Assume that we can differentiate a power series term by term. By taking the derivative of $f(x)$ with respect to $x$ and substituting the result into the differential equation (8.3), show that the equation

$$\sum_{k=1}^{\infty} ka_k x^{k-1} = \sum_{n=0}^{\infty} a_k x^k$$

must be satisfied in order for $f(x) = \sum_{k=0}^{\infty} a_k x^k$ to be a solution of the DE.

(b) Two series are equal if and only if they have the same coefficients on like power terms. Use this fact to find a relationship between $a_1$ and $a_0$.

(c) Now write $a_2$ in terms of $a_1$. Then write $a_2$ in terms of $a_0$.

(d) Write $a_3$ in terms of $a_2$. Then write $a_3$ in terms of $a_0$.

(e) Write $a_4$ in terms of $a_3$. Then write $a_4$ in terms of $a_0$.

(f) Observe that there is a pattern in (b)-(e). Find a general formula for $a_k$ in terms of $a_0$.

(g) Write the series expansion for $y$ using only the unknown coefficient $a_0$. From this, determine what familiar functions satisfy the differential equation (8.3). (Hint: Compare to a familiar Taylor series.)
Activity 8.27.

Determine the interval of convergence of each power series.

(a) \[ \sum_{k=1}^{\infty} \frac{(x - 1)^k}{3k} \]

(b) \[ \sum_{k=1}^{\infty} kx^k \]

(c) \[ \sum_{k=1}^{\infty} \frac{k^2(x + 1)^k}{4^k} \]

(d) \[ \sum_{k=1}^{\infty} \frac{x^k}{(2k)!} \]

(e) \[ \sum_{k=1}^{\infty} k!x^k \]
Activity 8.28.

Our goal in this activity is to find a power series expansion for \( f(x) = \frac{1}{1 + x^2} \) centered at \( x = 0 \).

While we could use the methods of Section 8.5 and differentiate \( f(x) = \frac{1}{1 + x^2} \) several times to look for patterns and find the Taylor series for \( f(x) \), we seek an alternate approach because of how complicated the derivatives of \( f(x) \) quickly become.

(a) What is the Taylor series expansion for \( g(x) = \frac{1}{1 - x} \)? What is the interval of convergence of this series?

(b) How is \( g(-x^2) \) related to \( f(x) \)? Explain, and hence substitute \(-x^2\) for \( x \) in the power series expansion for \( g(x) \). Given the relationship between \( g(-x^2) \) and \( f(x) \), how is the resulting series related to \( f(x) \)?

(c) For which values of \( x \) will this power series expansion for \( f(x) \) be valid? Why?
Activity 8.29.

Let $f$ be the function given by the power series expansion

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$ 

(a) Assume that we can differentiate a power series term by term, just like we can differentiate a (finite) polynomial. Use the fact that

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

to find a power series expansion for $f'(x)$.

(b) Observe that $f(x)$ and $f'(x)$ have familiar Taylor series. What familiar functions are these? What known relationship does our work demonstrate?

(c) What is the series expansion for $f''(x)$? What familiar function is $f''(x)$?
Activity 8.30.

Find a power series expansion for $\ln(1 + x)$ centered at $x = 0$ and determine its interval of convergence. (Hint: Use the Taylor series expansion for $\frac{1}{1 + x}$ centered at $x = 0$.)
Voting Questions

8.6.1 Consider the power series \( \sum_{n=1}^{\infty} \frac{(x-4)^n}{4^n} \). What values of \( x \) will make this series converge?

(a) This series converges for all values of \( x \).
(b) This series converges for all values of \( x \) between 0 and 8.
(c) This series converges for all values of \( x \) between -4 and 4.
(d) This series converges for all values of \( x \) between -8 and 0.
(e) This series diverges for all values of \( x \).

8.6.2 Consider the power series \( \sum_{n=1}^{\infty} \frac{(x-4)^n}{4^n} \). Will this series converge if \( x = 0 \) or if \( x = 8 \)?

(a) This series converges for both \( x = 0 \) and \( x = 8 \).
(b) This series does not converge for either \( x = 0 \) or \( x = 8 \).
(c) This series converges for \( x = 8 \) but does not converge for \( x = 0 \).
(d) This series converges for \( x = 0 \) but does not converge for \( x = 8 \).

8.6.3 Consider the power series \( \sum_{n=1}^{\infty} \frac{(3x)^n}{n^8} \). What values of \( x \) will make this series converge?

(a) This series converges for all values of \( x \).
(b) This series converges for all values of \( x \) between -3 and 3.
(c) This series converges for all values of \( x \) between 0 and 3.
(d) This series converges for all values of \( x \) between -1/3 and 1/3.
(e) This series diverges for all values of \( x \).

8.6.4 Consider the power series \( \sum_{n=1}^{\infty} \frac{(2x)^n}{n^7} \). Will this series converge if \( x = -1/2 \) or if \( x = +1/2 \)?

(a) This series converges for both \( x = -1/2 \) and \( x = +1/2 \).
(b) This series does not converge for either \( x = -1/2 \) or \( x = +1/2 \).
(c) This series converges for \( x = -1/2 \) but does not converge for \( x = +1/2 \).
(d) This series converges for \( x = +1/2 \) but does not converge for \( x = -1/2 \).

8.6.5 Consider the power series \( \sum_{n=1}^{\infty} \frac{(x-8)^n}{n(-6)^n} \). What values of \( x \) will make this series converge?

(a) This series converges for all values of \( x \).
(b) This series converges for all values of $x$ between 2 and 14.
(c) This series converges for all values of $x$ between -8 and 8.
(d) This series converges for all values of $x$ between 0 and 16.
(e) This series diverges for all values of $x$.

8.6.6 Consider the power series \( \sum_{n=1}^{\infty} \frac{(x-5)^n}{n(-3)^n} \). Will this series converge if $x = 2$ or if $x = 8$?

(a) This series converges for both $x = 2$ and $x = 8$.
(b) This series does not converge for either $x = 2$ or $x = 8$.
(c) This series converges for $x = 2$ but does not converge for $x = 8$.
(d) This series converges for $x = 8$ but does not converge for $x = 2$.

8.6.7 A power series converges when $x = 2.5, 2.7$ and 2.8, but diverges when $x = 2.1, 2.2$ and 2.9. Which of the following could be the point where the power series is centered?

(a) 2.3
(b) 2.6
(c) 2.7
(d) 2.8
(e) All of the above are possible.
(f) More than one but not all of the above are possible.
8.7 Functional DNA

Preview Activity 8.7. In this activity you’ll use mathematical DNA sequences to build approximations of functions. Use the GeoGebra applet found http://www.geogebratube.org/student/m204475 to do the plotting.

For each of the following:
Plot $T_0(x), T_1(x), T_2(x), T_3(x), \ldots$ one at a time using the GeoGebra applet linked above. Which function do you conjecture is being described by this sequence? Once you have your conjecture, use the GeoGebra applet to test your conjecture by plotting the functions on top of each other.

(a) Which well-known function is being described by the sequence

\[ \left\{ 0, 1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, 0, -\frac{1}{7!}, \cdots \right\} ? \]

To get you started the first few polynomials are written here for you.

\[
\begin{align*}
T_0(x) &= 0 \cdot 1 \\
T_1(x) &= 0 \cdot 1 + 1 \cdot x \\
T_2(x) &= 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \\
T_3(x) &= 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + \left(-\frac{1}{3!}\right) x^3 \\
T_4(x) &= 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + \left(-\frac{1}{3!}\right) x^3 + 0 \cdot x^4 \\
T_5(x) &= 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + \left(-\frac{1}{3!}\right) x^3 + 0 \cdot x^4 + \left(\frac{1}{5!}\right) x^5
\end{align*}
\]

(b) Which well-known function is being described by the sequence

\[ \left\{ 1, 0, -\frac{1}{2!}, 0, \frac{1}{4!}, 0, -\frac{1}{6!}, \cdots \right\} ? \]

(c) Which well-known function is being described by the sequence

\[ \left\{ 1, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \cdots \right\} ? \]
Activity 8.31.

(a) Consider the function \( f(x) = \sin(x) \) centered at \( x = 0 \). Write the Taylor sequence, Taylor series, and show the evolution of the sine function with several truncated Taylor polynomials.

(b) Consider the function \( f(x) = \sin(x) \) centered at \( x = \pi/4 \). Write the Taylor sequence, Taylor series, and show the evolution of the sine function with several truncated Taylor polynomials.

(c) Consider the function \( f(x) = \frac{1}{1-x} \) centered at \( x = 0 \). Write the Taylor sequence, Taylor series, and show the evolution of the sine function with several truncated Taylor polynomials.

(d) Consider the function \( f(x) = \frac{1}{1-x} \) centered at \( x = 2 \). Write the Taylor sequence, Taylor series, and show the evolution of the sine function with several truncated Taylor polynomials.

(e) It appears that parts (a) and (b) break the genetic analogy for Taylor series in that there are two completely different sequences describing the same function. How can this be possible?

(f) For the sine function and the exponential function (in the previous example) it appears that if the Taylor series is taken with more and more terms then the series more accurately describes more of the function. This is not true in parts (c) and (d) of this activity with the function \( f(x) = 1/(1 - x) \). Make a conjecture as to why this might be.
Activity 8.32.

The Fresnel sine function $S(x)$ is very important in the fields of optics, diffraction, and automobile design

$$S(x) = \int \sin(x^2) \, dx \quad \text{such that} \quad S(0) = 0.$$ 

The trouble is that there are no integration techniques that will give antiderivatives for $\sin(x^2)$. We use the Heredity principle to describe the Taylor series for $S(x)$ and therefore use the resulting polynomial to approximate and plot the function.

(a) Write the Taylor series for $\sin(x)$ centered at $x = 0$.

(b) Write the Taylor series for $\sin(x^2)$ by simply substituting $x$ with $x^2$ in the Taylor series for $\sin(x)$.

(c) Integrate each term in the Taylor series to arrive at a Taylor approximation for $S(x)$.

(d) Use graphing technology to plot of $S(x)$ on the domain $[-\pi, \pi]$. Experiment with the number of terms in the series to determine when your plot is accurate enough.
Activity 8.33.

Use the Heredity principle to find Taylor series for each of the following functions. Using graphing technology, make a plot of each Taylor series and compare with the plot of the original function.

(a) \( f(x) = xe^{x^2} \)

(b) \( f(x) = \frac{1}{1+x^2} \). Hint: \( x^2 = -(\frac{1}{x^2}) \) and then look back at Activity 8.31.

(c) \( f(x) = \int \cos(x^2) \, dx \) with \( f(0) = 0 \).

(d) \( f(x) = \int e^{-x^2} \, dx \) with \( f(0) = 0 \).
Activity 8.34.

Use the Heredity principle and some mathematical forensics to determine the solution to the differential equation. Create a plot of the solution.

(a) \( y' = 2y \) with \( y(0) = 1 \).

(b) \( y' = t^2 y \) with \( y(0) = 1 \).

(c) \( y'' - ty' - y = 0 \) with \( y(0) = 1 \) and \( y'(0) = 0 \). (Hint: You can make sense of the solution to this problem when you think of it as a spring-mass system)

(d) \( y'' + ty' + y = 0 \) with \( y(0) = 1 \) and \( y'(0) = -1 \). (Hint: You can make sense of the solution to this problem when you think of it as a spring-mass system)
Activity 8.35.

Use the GeoGebra applet (or another graphing utility) to empirically determine the radius of convergence for each of the following functions’ Taylor series with the given centers. 

http://www.geogebratube.org/student/m126820

(a) \( f(x) = \sqrt{x} \) centered at \( x = 1 \).

(b) \( f(x) = \sqrt{x} \) centered at \( x = 4 \).

(c) \( f(x) = \frac{1}{1-x} \) centered at \( x = 2 \).

(d) \( f(x) = \frac{1}{1-x} \) centered at \( x = 4 \).

(e) \( f(x) = x^{1/3} \) centered at \( x = 1 \).

(f) \( f(x) = x^{1/3} \) centered at \( x = 4 \).

(g) \( f(x) = e^{-x^2} \) centered at \( x = 0 \).

(h) \( f(x) = \frac{1}{1+x^2} \) centered at \( x = 0 \).

(i) In parts (a) - (f) what are the radii of convergence? Could you have guessed that before you made any plots?

(j) The functions in parts (g) and (h) look very similar at the outset but have very different radii of convergence. Use the Heredity principle to explain why.
Activity 8.36.

Determine where the power series converges. Only use the ratio test if it is absolutely necessary.

(a) \[ \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

(b) \[ \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k + 1)!} \]

(c) \[ \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \]

(d) \[ \sum_{k=0}^{\infty} \frac{(x/2)^k}{k!} \]

(e) \[ \sum_{k=0}^{\infty} \frac{x^k}{k^2} \]

(f) \[ \sum_{k=0}^{\infty} k!x^k \]

(g) \[ \sum_{k=0}^{\infty} \frac{x^k}{(2k)!} \]

(h) \[ \sum_{k=0}^{\infty} \frac{k^2(x + 1)^k}{4^k} \]

(i) Write the Taylor series for \( f(x) = \ln(x) \) centered at \( x = 1 \) and determine the radius of convergence.